INTERNATIONAL JOURNAL OF PURE MATHEMATICS
DOI: 10.46300/91019.2026.13.2 Volume 13, 2026

Specific Approach of the Noetherization Theory For an Integro-Differential Operator A in the Space $C_{-1}^{(m)}[-1,1]$

Abdourahman Haman Adji[1, *], Shankishvili Lamara Dmitrievna[2]
[1]Department of Mathematics and Computer Sciences, Faculty of Sciences,
University of Ngaoundéré,
PO.BOX 454. Ngaoundéré,
CAMEROON

[2]Department of Mathematics,
Georgian Technical University,
Kostova Street, Tbilisi 0171, Georgia Tbilisi,
GEORGIA

[*]*Corresponding Author*

Received: March 9, 2026. Revised: June 11, 2026. Accepted: August 18, 2026. Published: November 12, 2026.

Abstract—In this article, our primary objective is to construct a noetherian theory for a singular linear integro-differential operator. This operator is defined by an integral equation of the third kind, the principal part of which is expressed as a higher-order differential operator. The choice of a methodology requiring specific approaches proved justified, as it stems from solving the linear differential equation under study—featuring a specific, well-known non-homogeneous term—thereby allowing for the systematic identification of solvability conditions. In this research, we employ integration techniques to establish the noetherian property of the operator, while highlighting the solvability conditions for the integral equation of the third kind that defines this noetherian integro-differential operator. Ultimately, we establish the noetherian properties for both the principal part operator and its associated, subsequently developing the noetherian theory for the operator under study in parallel with that of its associated. A concrete illustrative example of the noetherian theory, applied to an integro-differential operator *A*, is presented at the conclusion of the work.

2020 Mathematics Subject Classification: 34A25, 45A05, 45B05, 46A32, 46A50, 47A52.

Keywords—noetherian theory, third kind integral equation, singular linear integro-differential operator, deficient numbers, index of the operator, associated operator and associated space.

I. INTRODUCTION

Scientific research in mathematics carried out by several mathematicians in the field of the theory of integro-differential operators, defined by integral equations of the third kind in certain well-chosen distributional functional spaces, is the subject of the content of various scientific publications which have been mentioned both in scientific works and monographic works as well as in several bibliographic references. For example and for full details see [1], [2], [3], [4], [5], [6], [7] and [8].

And for this reason, as is well known, the noetherization of integro-differential operators defined analytically by certain types of linear integral equations of the third kind is widely illustrated in the investigations conducted, the results of which are the subject scientific articles also published in certain international mathematical journals and reviews. [9], [10] and [11]. In particular, the construction of the noetherian theory of integro-differential operators defined by integral equations in general occupies an important place not only to establish the said theory but also to establish the conditions of solvability of the said integral equations defining these operators.

However, it should be noted that the main obstacle encountered when investigating the solvability of integral equations of the third kind during the noetherization process for integro-differential operators expressed analytically by these equations actually stems from the difficulty in choosing the approaches and methods necessary to achieve the desired objective. Among others, let us mention a remarkable fact that during the process of noetherization of integro-differential operators the effective use of different methods namely: that of normalization, that of hypersingular integrals as well as

that of approximate inverse operators allowing to approach the research on the integral equations of the third kind in the specific functional spaces of distributions and continuous functions. See [12], [13], [14] and [15]. In a specific case, it is worth noting that the various ideas related to the aforementioned various approaches and methods systematically allow to clearly define and pose the problem during the investigation process, relative to the integral equations of the first kind, or non-Fredholmian equations of the second kind, thus defining the previously considered operators.

We will also rightly mention that, among others, eminent researchers on their side have investigated the inverse problem of the spectral analysis of the theory of linear multigroup neutron transport in plane geometry in their scientific works.

All this being said, then specifically in this work to be carried out, we consider the following integro-differential operator defined by a singular linear integral equation of the third kind of the form:

$$(A\varphi)(x) = x^p\varphi^{(m)}(x) + \int_{-1}^{1} k(x,t)\,\varphi(t)\,dt = f(x); x \in [-1,1], \tag{1}$$

where $m > 1$, $\varphi(x) \in C_{-1}^{(m)}[-1,1], f(x) \in C_0^{\{p\}}[-1,1]$ and $k(x,t) \in C_0^{\{p\}}[-1,1]\,X\,C\,[-1,1]$.

The establishment of the noetherian property of the integro-differential operator A defined by (1) will be formalized systematically through the investigation of the questions of the solvability of the m higher order linear differential equation $(L\varphi)(x) = x^p\varphi^{(m)}(x) = f(x)$, which will lead to the determination of the noetherity of the initial operator L and by ricochet that of the initial operator A.

Rigorous proofs are to be provided regarding the compactness of the integral operators, as well as the concept and principle of associated operators within the article while taking into consideration that when a certain noetherian operator is perturbed by a completely continuous operator, the noetherian property of the operator and the index of the operator are preserved.

The content of this work of investigation of the construction of the noetherian theory of the considered operator A is as follows: first, we make a detailed presentation in section II of the preliminaries related to the different concepts and notions of the famous noetherian theory, necessary for the execution of the related research. The main important results of the work carried out are stated and formulated through different lemmas and theorems contained in section III. And to complete this work, we draw up a summary of synthesis, descriptive of all the investigations carried out located in section IV, entitled Conclusion then, followed by some orientations and recommendations for the continuation and realization of future scientific work.

II. PRELIMINARIES

We use the different concepts and notions related to the noetherian theory of integro-differential operators, some of the results of which have been presented in our previous published works and which we present here once again, to allow a good understanding of the work carried out.

We also recall the notions of Taylor's sense derivatives including that of the third kind of linear Fredholm integral equation, which have been widely studied and used by many researchers in their work related to the noetherian theory.

First of all let us move to the following concept.

A) Noetherian operator.

Definition 1. Let X, Y be Banach spaces, $A \in l(X,Y)$ a linear operator. The quotient space $coker\,A = Y/imA$ is called the cokernel of the operator A. The dimensions $\alpha(A) = \dim ker A, \beta(A) = \dim coker A$ are called the nullity and the deficiency of the operator A, respectively. If at least one of the numbers $\alpha(A)$ or $\beta(A)$ is finite, then the difference $Ind\,A = \alpha(A) - \beta(A)$ is called the index of the operator A.

Definition 2. Let X, Y be Banach spaces, $A \in l(X,Y)$ is said to be normally solvable if it possesses the following property: the equation $Ax = y\,(y \in Y)(y \in Y)$ has at least one solution $x \in D(A)$ ($D(A)$ is the domain of A) if and only if $< y, f >= 0\ \forall\ f \in (im\,A)^{\perp}$ holds.

We recall that by the definition of the adjunct operator $(im\,A)^{\perp} = \ker A^*$ and that the operator A is normally solvable if and only if its image space imA is closed.

Definition 3. A closed normally solvable operator A is called a noetherian operator if its index is finite.

By the way, we briefly review this important notions of Taylor derivatives which is widely used when constructing noetherian theory of the considered operator A.

Definition 4. A Continuous function $\varphi(x)\,\epsilon\,C[-1,1]$ admits at the point $x = 0$ Taylor derivatives up to the order $p \in \mathbb{N}$ if there exists recurrently for $k = 1,2,\dots,p$, the following limits :

$$\varphi^{\{k\}}(0) = k!\ \lim_{x\to 0} x^{-k}\left[\varphi(x) - \sum_{j=0}^{k-1}\frac{\varphi^{\{j\}}(0)}{j!}x^j\right]. \tag{2}$$

The class of such functions $\varphi(x)$ is denoted $C_0^{\{p\}}[-1,1]$.

Next let us move to the following part.

Let $C^m[-1,1], m \in \mathbb{Z}_+$, noted the Banach space of continuous functions on $[-1,1]$, having continuous derivatives up to order m, for which the norm is defined as following:

$$\|\varphi(x)\|_{C^m[-1,1]} = \sum_{j=0}^{m} \max_{-1\le x\le 1}\left|\varphi^{(j)}(x)\right| \tag{3}$$

So, that we can consider $\varphi^{\{k\}}(0)$ are defined for all $k = 1,2, \dots, p$.

We define $C_0^{\{p\}}[-1,1]$ as a subspace of continuous functions, having finite Taylor derivatives up to order $p \in \mathbb{Z}_+$; and when $p = 0$, we put $(C_0^{\{p\}}[-1,1] = C_0^{\{0\}}[-1,1] = C[-1,1])$. Let us also define a linear operator N^k on the space $C_0^{\{p\}}[-1,1]$ by the formula:

$$(N^k\varphi)(x) = \frac{\varphi(x) - \sum_{j=0}^{k-1} \frac{\varphi^{\{j\}}(0)}{j!} x^j}{x^k}, k = 1,2, \dots, p. \quad (4)$$

One can easily verify the property $N^k = N^{k_1} N^{k-k_1}, 0 \le k_1 \le k, k, k_1 \in \mathbb{Z}_+$, where we put $N^0 = I$.

Definition 5. The operator N^p is called characteristical operator of the space $C_0^{\{p\}}[-1,1]$.

Remark: The sense of the previous definition can be seen from the verification of the following lemma.

Lemma 1. A function $\varphi(x)$ belongs to the space $C_0^{\{p\}}[-1,1]$ if and only if, the following representation

$$\varphi(x) = x^p \phi(x) + \sum_{k=0}^{p-1} \alpha_k x^k \quad (5)$$

holds with the function $\phi(x) \in C[-1,1]$, and α_k being constants.

To prove Lemma 1 it is enough to observe that (5) implies that the Taylor derivatives of $\varphi(x)$ up to the order p exists, and more $\varphi^{\{k\}}(0) = k!\, \alpha_k, k = 0,1,2, \dots, p-1$, $\varphi^{\{0\}}(0) = p!\, \phi(0)$ with $\phi(x) = (N^k\varphi)(x)$. Conversely, if $\varphi(x)$ belongs to $C_0^{\{p\}}[-1,1]$, and we define $\phi(x) = (N^k\varphi)(x)$ with $\alpha_k = \frac{\varphi^{\{k\}}(0)}{k!}, k = 0,1,2, \dots, p-1$, then the representation (5) holds. From Lemma 1, it follows that for $\varphi(x) \in C_0^{\{p\}}[-1,1]$ the equality

$$\varphi(x) = x^p (N^k\varphi)(x) + \sum_{k=0}^{p-1} \frac{\varphi^{\{k\}}(0)}{k!} x^k, \quad (6)$$

is valid.

Consequently, the linear operator N^p establishes a relation between the spaces $C_0^{\{p\}}[-1,1]$ and $C[-1,1]$. The space $C_0^{\{p\}}[-1,1]$ with the norm

$$\|\varphi\|_{C_0^{\{p\}}[-1,1]} = \|N^p\varphi\|_{C[-1,1]} + \sum_{k=0}^{p-1} |\varphi^{\{k\}}(0)|, \quad (7)$$

becomes a Banach space one.

Let note also that we can define the previous norm by the following way:

$$\|\varphi\|_{C_0^{\{p\}}[-1,1]} = \|N^p\varphi\|_{C[-1,1]} + \sum_{k=0}^{p-1} |\alpha_k| = \|\phi(x)\|_{C[-1,1]} + \sum_{k=0}^{p-1} |\alpha_k|$$

.

Sometimes it is comfortable and suitable to consider as norm in the space $C_0^{\{p\}}[-1,1]$ the equivalent norm defined as follow:

$$\|\varphi\|_{C_0^{\{p\}}[-1,1]} = \sum_{j=0}^{p} \|N^j\varphi\|_{C[-1,1]} \quad (8)$$

We can also note a very useful and clearly helpful next inequality:

$$\|\varphi\|_{C[-1,1]} \le \|N^p\varphi\|_{C[-1,1]} + \sum_{j=0}^{p-1} |\varphi^{\{j\}}(0)| = \|\varphi\|_{C_0^{\{p\}}[-1,1]} \quad (9)$$

Therefore, it is obvious to see that $\|\varphi\|_{C[-1,1]} \le \|\varphi\|_{C_0^{\{p\}}[-1,1]}$.

Finally, note that from the Lemma 1 it follows the following fact, if $\varphi(x) \in C[-1,1]$, then $x^p\varphi(x) \in C_0^{\{p\}}[-1,1]$. This assertion may be generalized as follows.

Lemma 2. Let $p \in \mathbb{N}, s \in \mathbb{Z}_+$. If $\varphi(x) \in C_0^{\{s\}}[-1,1]$ then, $x^p\varphi(x) \in C_0^{\{p+s\}}[-1,1]$, and the formula holds

$$\left(x^p\varphi(x)\right)^{\{j\}}(0) = \begin{cases} 0, & j = 0,1, \dots, p-1, \\ \frac{j!}{(j-p)!}\varphi^{\{j-p\}}(0), & j = p, \dots, p+s. \end{cases} \quad (10)$$

Proof. Note that a stronger assumption on the function $\varphi(x)$, such that $\varphi(x) \in C_0^{\{p+s\}}[-1,1]$ would allow us to easily prove the lemma just by applying Leibniz formula.

For $s = 0$ the statement has already been proved above, so $x^p\varphi(x) \in C_0^{\{p\}}[-1,1]$,and $\left(x^p\varphi(x)\right)^{\{j\}}(0) = 0, j = 0, \dots, p-1$ and $\left(x^p\varphi(x)\right)^{\{p\}}(0) = p!\, \varphi(0)$. Now let us prove that $\left(x^p\varphi(x)\right)^{\{j\}}(0) = \frac{j!}{(j-p)!}\varphi^{\{j-p\}}(0), j = p+1, \dots, p+s$.

Since the derivatives are defined recurrently, and (10) is true for $j = p$, then it is sufficient to verify the passage from j to $j+1$. We have:

$$\left(x^p\varphi(x)\right)^{\{j+1\}}(0) = (j+1)! \lim_{x \to 0} \frac{x^p\varphi(x) - \sum_{l=p}^{j} \frac{x^l}{(l-p)!}\varphi^{\{l-p\}}(0)}{x^{j+1}} \quad (11)$$

$$= (j+1)! \lim_{x \to 0} \frac{\varphi(x) - \sum_{l=0}^{j-p} \frac{x^l \varphi^{\{l\}}(0)}{l!}}{x^{j+1-p}} = \frac{(j+1)!}{(j+1-p)!}\varphi^{\{j+1-p\}}(0) \quad (12)$$

.

Lemmas 1 and 2 imply the next important lemma.

Lemma3.Let
$f(x) \in C_0^{\{p\}}[-1,1], p \in \mathbb{N}$ and $f(0) = \ldots = f^{\{r-1\}}(0) = 0, 1 \leq r \leq p.$
Then $\frac{f(x)}{x^r} \in C_0^{\{p-s\}}[-1,1]$.

Definition 6. The linear Fredholm integral equation of the third kind is an equation of the form $A_n\varphi = f$ defined analytically by the expression

$$A_n\varphi = g_n\varphi(x) - \int_a^b k(x,t)\varphi(t)\,dt; \qquad (13)$$

where f is a given function of $x \in [a,b]$,$\varphi(x)$ is the unknown function of $x \in [a,b]$.

Here, $g_n(x) = \prod_{k=1}^n (x - x_k)$ is a given function of the variable $x \in [a,b]$ with $x_k \in]a,b[$, and the kernel $k(x,t)$ is a given function of two independent variables $(x,t) \in [a,b]X[a,b]$.

Importantly, let us recall that also in transport theory, the natural class of solutions of the integral equation (13) is a class of distributional functions in the following form: $\varphi(x) = y(x) + \sum_{k=1}^n \alpha_k P\frac{1}{x-x_k} + \sum_{k=1}^n \omega_k \delta(x - x_k)$, where $y(x) \in C[a,b]$, $\delta(x)$ is the zero-centered Dirac delta function, and P indicates that Cauchy's principal value is used for integration of $\frac{1}{x-x_k}$.

Since the first works of mathematicians Hilbert and Picard dedicated to the integral equation of the third kind, it is worth noting that several investigations whose scientific results are published in various articles in internationally renowned mathematical journals and newspapers have been devoted to this vast field. In their related research, researchers Bart and Warnock in turn studied the equation (13) in the distributional space of distributions defined in the form: $\varphi(x) = y(x) + \sum_{k=1}^n \omega_k \delta(x - x_k), \quad y(x) \in C[a,b]$, and were able to demonstrate that under the fulfillment of the condition $det\left(k(x_i - x_j)\right)_{1\leq i,j\leq n} \neq 0$ three of Fredholm's theorems hold and are valid. Following the work of Bart and Warnock, the researcher Sukavanam studied the equation (13) with some additional hypotheses on such kernels of the integral operator.

B) Associated operators and associated spaces.

Definition 7. The Banach space $\mathrm{E}' \subset \mathrm{E}^*$ is called associated space with the space E, if $|(\mathrm{f},\varphi)| \leq \mathrm{c}\|\mathrm{f}\|_{\mathrm{E}'}\|\varphi\|_{\mathrm{E}}$ for every $\varphi \in \mathrm{E}, \mathrm{f} \in \mathrm{E}'$.

We note that the initial space E can be considered associated with the space E'. Moreover, the norm $\|\mathrm{f}\|_{\mathrm{E}'}$ is not obliged to be equivalent to the norm $\|\mathrm{f}\|_{\mathrm{E}^*}$.

Let be noted $\mathcal{L}(\mathrm{E}_1, \mathrm{E}_2)$ the Banach algebra of all linear bounded operators from E_1 into E_2.

Definition 8. Let $\mathrm{E}_j, j = 1,2$ two Banach spaces and E'_j their associated spaces. The operators $\mathrm{A} \in \mathcal{L}(\mathrm{E}_1, \mathrm{E}_2)$ and $\mathrm{A}' \in \mathcal{L}(\mathrm{E}'_2, \mathrm{E}'_1)$ are called associated, if $(\mathrm{A}'\mathrm{f},\varphi) = (\mathrm{f},\mathrm{A}\varphi)$ for all $\mathrm{f} \in \mathrm{E}'_2$ and $\varphi \in \mathrm{E}_1$.

It seems that, we can formalize the noetherity property in terms of associated operator and associated space following theoretical researches related to this field undertaken by the Georgian mathematician Duduchava R.V.

Lemma 4 Let $\mathrm{E}_j, j = 1,2$ two Banach spaces and E'_j their associated spaces and more, let $\mathrm{A} \in \mathcal{L}(\mathrm{E}_1, \mathrm{E}_2)$ and $\mathrm{A}' \in \mathcal{L}(\mathrm{E}'_2, \mathrm{E}'_1)$ be associated noetherian operators and finally,

$$\alpha(\mathrm{A}) = -\alpha(\mathrm{A}').$$

Then, for the solvability of the integral equation $\mathrm{A}\varphi = \mathrm{f}$ it is necessary and sufficient that $(\mathrm{f},\psi) = 0$ for all solutions of the homogeneous associated equation $\mathrm{A}'\psi = 0$.

Next, let us move to the presentation of the general important results of the work in the following section.

III. MAIN RESULTS.

Main part of the operator $\boldsymbol{A}$.

In this section, we first address the question of constructing the noetherian theory of the principal part of our integro-differential operator A, which we denote by the operator L.

To this end, we focus our investigations on the model of the m^{th} higher-order linear differential equation to be studied, expressed by formula (1) by highlighting separately the differential operator L defined by the following formula.

$$(L\varphi)(x) = x^p\varphi^{(m)}(x) = f(x), x \in [-1,1]. \qquad (14)$$

We recall that $f(x) \in C_0^{\{p\}}[-1,1]$ and has the representation

$$f(x) = x^p g(x) + \sum_{k=0}^{p-1} c_k x^k, \qquad (15)$$

where $g(x) \in C[-1,1], c_k = \frac{f^{\{k\}}(0)}{k!}$ and $g(x) = (N^p f)(x)$.

It is obvious that the homogeneous differential equation

$$x^p\varphi^{(m)}(x) = 0 \qquad (16)$$

in the space $C_{-1}^{(m)}[-1,1]$ has a solution in the form of a polynomial type of the $(m-1)$ degree so that $\varphi(x) = P_{m-1}(x) = \sum_{k=0}^{m-1} c_k x^k$ and, for that by virtue of the conditions $\varphi^{(k)}(-1) = 0, k = 0, \ldots, m-1$ in the space $C_{-1}^{(m)}[-1,1]$, the unique solution of the equation (16) will be $\varphi(x) = 0$.

Next we consider the nonhomogeneous equation (14) with respect to the fact that $f(x) \in C_0^{\{p\}}[-1,1]$ i.e after division of the right hand side by the term x^p , we reach the following result:

$$\varphi^{(m)}(x) = g(x) + \frac{1}{x^p}\sum_{k=0}^{p-1} c_k x^k. \qquad (17)$$

For the solvability of this equation in the class of continuous functions it is necessary to require that all coefficients $c_k = 0,\ k = 0, \ldots, p-1$ or the same as $\left(\delta^{\{k\}}(x), f(x)\right) = 0$ under $k = 0, \ldots, p-1$. In this case $\varphi^{(m)}(x) = g(x)$. integrating once with respect to the fact that $\varphi^{(m-1)}(-1) = 0$, we have the following result:

$$\varphi^{(m-1)}(x) = \int_{-1}^{x} g(t)dt, \quad (19)$$

from where it follows

$$\varphi(x) = \int_{-1}^{x} du \int_{-1}^{u} dv \int_{-1}^{v} dw \dots \int_{-1}^{s} g(t)dt = \frac{1}{(m-1)!}\int_{-1}^{x}(x-t)^{m-1} g(t)dt \quad . \quad (20)$$

Therefore finally we obtain that the equation (14) has a unique solution defined by the next formula:

$$\varphi(x) = \frac{1}{(m-1)!}\int_{-1}^{x}(x-t)^{m-1} g(t)dt \quad (21)$$

under accomplishment of p −conditions of the form $\left(\delta^{\{k\}}(x), f(x)\right) = 0$ under $k = 0, \dots, p-1$.

In this case the deficient numbers of the operator L has the form $(0, p)$ i.e $\alpha(L) = 0$, $\beta(L) = p$ and the index $\chi(L) = -p$.

Next our further goal is to show that the obtained resolvability conditions are nothing other than the orthogonality conditions of the homogeneous associated equation in the associated space. But before going to this let us introduce the following space $P_m^1 = P_m^1[-1,1]$.

Let designate through $P_m^1[-1,1]$ the space of distributional functions onto the space of test functions $C_0^{\{p\}}[-1,1]$ such that $\Psi(x) \in P_m^1[-1,1]$ if $\Psi(x)$ has the following presentation:

$$\Psi(x) = \frac{z(x)}{x^p} + \sum_{k=0}^{p-1}\beta_k \delta^{\{k\}}(x), \quad (22)$$

where $z(x) \in C_0^{\{p\}}[-1,1] \cap C_1^{(m)}[-1,1]$, so that $z(x) = x^p h(x) + \sum_{k=0}^{p-1} c_k x^k$ and $z^{(k)}(1) = 0$, $k = 0, \dots, m-1$, where $h(x) \in C[-1,1]$.

Lemma 5. The spaces $C_{-1}^{(m)}[-1,1]$ with $C[-1,1]$ and $P_m^1[-1,1]$ with $C_0^{\{p\}}[-1,1]$ form two pairs of associated Banach spaces.

Now let us prove that the conditions $\left(\delta^{\{k\}}(x), f(x)\right) = 0$ under $k = 0, \dots, p-1$ are the conditions of orthogonality to the solutions of the homogeneous associated equation.

For this, first of all let us define the associated operator $L' : P_m^1[-1,1] \longrightarrow C[-1,1]$ to the operator L by the following way:

$$(L'\Psi)(x) \equiv (-1)^m (x^p \Psi(x))^{(m)} \quad (23)$$

Let us verify that of course, the operators L and L' are associated operators so that it takes place:

$$(L\varphi, \Psi) = (\varphi, L'\Psi) \ \forall \varphi \in C_{-1}^{(m)}[-1,1], \ \forall \Psi \in P_m^1[-1,1]. \quad (24)$$

Infact let us calculate the left part first of all:

$$(L\varphi, \Psi) = \left(x^p \varphi^{(m)}(x), \Psi(x)\right) = \left(x^p \varphi^{(m)}(x), \frac{z(x)}{x^p} + \sum_{k=0}^{p-1}\beta_k \delta^{\{k\}}(x)\right) = \left(\varphi^{(m)}(x), z(x)\right) = \int_{-1}^{1} \varphi^{(m)}(x) \cdot z(x)\, dx = \varphi^{(m-1)}(x) z(x)\Big|_{-1}^{1} - \int_{-1}^{1} \varphi^{(m-1)}(x) \cdot z'(x)\, dx = -\left(\varphi^{(m-1)}(x), z'(x)\right), \quad (25)$$

where it has been considered that $\varphi^{(m-1)}(-1) = z(1) = 0$. Continuing integration by part we reach the following relationship:

$$\left(\varphi^{(m)}(x), z(x)\right) = -\left(\varphi^{(m-1)}(x), z'(x)\right) = (-1)^m \left(\varphi(x), z^{(m)}(x)\right). \quad (26)$$

In the other side we have:

$$(\varphi(x), L'\Psi) = \left(\varphi(x), (-1)^m (x^p \Psi(x))^{(m)}\right) = (-1)^m \left(\varphi(x), z^{(m)}(x)\right). \quad (27)$$

Comparing (25) and (27) we conclude that $(L\varphi, \Psi) = (\varphi, L'\Psi) \quad \forall \varphi \in C_{-1}^{(m)}[-1,1]$, $\forall \Psi \in P_m^1[-1,1]$, and this means the operators L and L' are associated operators.

Lemma 6. The operator $L: C_{-1}^{(m)}[-1,1] \rightarrow C_0^{\{p\}}[-1,1]$ is noetherian operator and has a deficient numbers $(0, p)$. The nonhomogeneous equation $L\varphi = f$ is solvable under the accomplishment of p − conditions for orthogonality $\left(\delta^{\{k\}}(x), f(x)\right) = 0$ under $k = 0, \dots, p-1$, to all solutions to the homogeneous associated equation $L'\Psi = 0$ in the associated space P_m^1 .

Under the accomplishment of these conditions, the non-homogeneous equation has a unique solution defined by the formula (21).

Next we consider the homogeneous associated equation

$$(L'\Psi)(x) \equiv (-1)^m \left(x^p \Psi(x)\right)^{(m)} = 0 \quad (28)$$

in the space $P_m^1[-1,1]$. Setting the expression of $\Psi(x)$ in (27) we arrive to the following equation: $z^{(m)}(x) = 0$, that gives $z(x) = 0$ by virtue of the conditions $z^{(k)}(1) = 0$, $k = 0, \dots, m-1$.

Consequently the solution $\Psi(x)$ of the homogeneous associated equation has the following form:

$$\Psi(x) = \sum_{k=0}^{p-1}\beta_k \delta^{\{k\}}(x), \quad (29)$$

where β_k −are arbitrary constants.

Remark that the Dirac delta functions $\delta^{\{k\}}(x)$ are linearly independent. So that, the previously obtained conditions

$\left(\delta^{\{k\}}(x), f(x)\right) = 0$ under $k = 0, \dots, p-1$ for the solvability of the non-homogeneous equation can be interpreted as conditions for the orthogonality of the right-hand side $f(x)$ to all the solutions of the homogeneous associated equation in the associated space.

Now consider the associated nonhomogeneous equation defined by the operator L' of the following form:

$$(L'\Psi)(x) \equiv (-1)^m \left(x^p \Psi(x)\right)^{(m)} = h(x), \tag{30}$$

where $h(x) \in C[-1,1]$.

Using as previously upstairs the representation $\Psi(x)$ from $P_m^1[-1,1]$ for the solutions, rewrite the equation (28) in the next form:

$$(-1)^m z^{(m)}(x) = h(x) \text{ or } z^{(m)}(x) = (-1)^m h(x), \tag{31}$$

from where integrating m −times and taking into consideration that $z^{(k)}(1) = 0, k = 0, \dots, m-1,$ we obtain:

$$z^{(m-1)}(x) = (-1)^m \int_x^1 h(t)dt \tag{32}$$

Consequently

$$z(x) = \frac{(-1)^m}{(m-1)!} \int_x^1 (x-t)^{m-1} h(t)dt. \tag{33}$$

Therefore, the solution $\Psi(x)$ of the equation (28) has the following form:

$$\Psi(x) = \frac{\frac{(-1)^m}{(m-1)!} \int_x^1 (x-t)^{m-1} h(t)dt}{x^p} + \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x), \tag{34}$$

with β_k arbitrary constants.

Consequently the operator $L': P_m^1[-1,1] \longrightarrow C[-1,1]$ is noetherian with the deficient numbers $(p, 0)$ i.e $\alpha(L') = p$, $\beta(L') = 0$ and with the index $\chi(L') = p$.

At the end of these previous analyses carried out, let us state at this stage the main result obtained in the form of a lemma.

Lemma 7. The operator $L': P_m^1[-1,1] \longrightarrow C[-1,1]$ is noetherian operator and has deficient numbers $(p, 0)$. The nonhomogeneous equation $L'\Psi = h$ has the solution defined by the following formula:

$$\Psi(x) = \frac{\frac{(-1)^m}{(m-1)!} \int_x^1 (x-t)^{m-1} h(t)dt}{x^p} + \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x), \tag{35}$$

with β_k arbitrary constants $, k = 0, \dots, m-1,$ in the associated space $P_m^1[-1,1]$.

Theorem 1 Let the kernel $k(x,t) \in C_0^{\{p\}}[-1,1] \,\mathrm{X}\, \mathrm{C}[-1,1]$, then the operator K of the form $(K\varphi)(x) = \int_{-1}^1 k(x,t)\,\varphi(t)dt, x \in [-1,1]$ is completely continuous operator from $C_{-1}^{(m)}[-1,1]$ into $C_0^{\{p\}}[-1,1]$.

Proof. Let $\varphi(t) \in C_{-1}^{(m)}[-1,1]$. Let's verify at the beginning that $(K\varphi)(t) \in C_0^{\{p\}}[-1,1]$. Using the presentation of the kernel $k(x,t)$ in the following form:

$$k(x,t) = \sum_{k=0}^{p-1} c_k(t)\, x^k + x^p \widehat{K_1}(x,t) \tag{36}$$

where $\widehat{K_1}(x,t)$ is continuous with respect to the set of variables $x, t \in [-1,1]$ and $c_k(t) \in \mathrm{C}\,[-1,1]$.

In other words, $k(x,t)$ has a Taylor derivative by the variables x at the point $(0,t)$ for each $t \in [-1,1]$.

We have $(K\varphi)(x) = \sum_{k=0}^{p-1} \widehat{c_k}\, x^k + x^p \Omega(x)$ where it is designated

$$\widehat{c_k} = \int_{-1}^1 c_k(t)\,\varphi(t)dt, \text{ and}$$
$$\Omega(x) = \int_{-1}^1 \widehat{K_1}(x,t)\ \varphi(t)dt. \tag{37}$$

Considering that $\Omega(x) \in \mathrm{C}\,[-1,1]$ we obtain $(K\varphi)(x) \in C_0^{\{p\}}[-1,1]$. From ( ) it is easy to receive the following approximation

$$\|K\varphi\|_{C_0^{\{p\}}[-1,1]} \leq c \|\varphi\|_{\mathrm{C}\,[-1,1]} \tag{38}$$

where $c = 2\left(M + \sum_{k=0}^{p-1} M_k\right)$,
$M_k = \max_t |c_k(t)|$ and
$M_k = \max_t |c_k(t)|$ and
$M = \max_{x,t} \left|\widehat{K_1}(x,t)\right|$ (39)

Finally let us proof that the operator K maps any bounded set $M_0 = \{\varphi\} \subset \mathrm{C}\,[-1,1]$ into a relatively compact set in $C_0^{\{p\}}[-1,1]$. To do this, we will use the following compactness criterion in the space $C_0^{\{p\}}[-1,1]$.

Lemma 8. The set $M \subset C_0^{\{p\}}[-1,1]$ into a relatively compact set in the space $C_0^{\{p\}}[-1,1]$ if and only if when: a) the set M is bounded; b) A family of continuous functions $N^p M$ is equicontinuous on a closed interval $[-1,1]$.

Proof. The necessary of the condition a) immediately follows from the boundedness of relatively compact sets. As for condition (b), a bounded operator $N^p: C_0^{\{p\}}[-1,1] \longrightarrow \mathrm{C}[-1,1]$ maps a relatively compact set M into $C_0^{\{p\}}[-1,1]$ to a relatively $N^p M$ compact set into $\mathrm{C}[-1,1]$, and thus (b) follows from the Arzelà criterion.
Sufficiency of evidence. Using the condition a) and the equivalent norm in the space $C_0^{\{p\}}[-1,1]$ in the following form

$$\|\varphi\|^1_{C_0^{\{p\}}[-1,1]} = \sum_{j=0}^{p} \left\| N^j \varphi \right\|_{\mathrm{C}[-1,1]} \quad (40)$$

we see that

$$|f(t)| \le r, \quad t \in [-1,1], \ \left|f^{\{j\}}(0)\right| \le r, j = 1, \dots, p \quad (41)$$

for all $f \in M$. By virtue of the obvious equality

$$\|\varphi\|_{C_0^{\{p\}}[-1,1]} = \| N^p \varphi\|_{\mathrm{C}[-1,1]} + \sum_{k=0}^{p-1} \left|\varphi^{\{k\}}(0)\right| \quad (42)$$

with respect to the equicontinuity of $(N^p f)(x), f \in M$, equicontinuity follows from the uniform continuity of power functions $x^j, j = 0, \dots, p$ and uniform boundedness of $\left|f^{\{j\}}(0)\right|, \ j = 1, \dots, p$ follows of $f(x) \in M$.

Therefore by the virtue of the Arzela criterion M relatively compact set into $\mathrm{C}[-1,1]$, and it remains with respect to the uniform boundedness of $\left|f^{\{j\}}(0)\right|, \ j = 1, \dots, p$ and Bolzano lemma follows the relative compactness into $C_0^{\{p\}}[-1,1]$, and which is what was required to be prove.

Continuing the proof of the theorem let us use this criterion. As M_0 is a bounded set, then by virtue of (38) it follows that and $M = K(M_0) \subset C_0^{\{p\}}[-1,1]$ also is bounded set. It remains to verify that the family of the functions

$$h = N^p K\varphi \in N^p(M) = N^p K M_0$$

where it is noted $h(x) = \int_{-1}^{1} \widehat{K_1}(x,t)\ \varphi(t)dt$ is equicontinuous. Approximating the difference

$$|h(x_1) - h(x_2)| = 2\left(\max_{x_1,x_2,t} \left|\widehat{K_1}(x_1,t) - \widehat{K_1}(x_2,t)\right|\right) \| \varphi\|_{\mathrm{C}[-1,1]}. \quad (43).$$

It remains to consider that the function $\widehat{K_1}(x,t)$ is equicontinuous by the variable x and $t \in [-1,1]$. This ends the proof of the theorem 1.

Next step let us consider the associated operator of the following form:

$$(K'\Psi)(x) = \int_{-1}^{1} k(t,x)\, \Psi(t)dt, x \in [-1,1] \quad (44)$$

The following theorem gives the condition of it compactness from the space $P_m^1[-1,1]$ into $C[-1,1]$.

Theorem 2 Let the kernel $k(t,x) \in C_0^{\{p\}}[-1,1] \,\mathrm{X}\, \mathrm{C}[-1,1]$, then the operator K' of the form $(K'\Psi)(x) = \int_{-1}^{1} k(t,x)\, \Psi(t)dt, x \in [-1,1]$ is completely continuous operator from $P_m^1[-1,1]$ into $C[-1,1]$.

Proof: For the functions $\Psi(x)$ from the space $P_m^1[-1,1]$ by virtue of the formula (21) we obtain the following representation

$$(K'\Psi)(x) = \int_{-1}^{1} k(t,x)\frac{z(t)}{t^p}dt + \sum_{k=0}^{p-1}(-1)^J \beta_j K_1^{\{j\}}(0,x), \quad (45)$$

where $K_1^{\{j\}}(0,x)$ designate the Taylor derivative of the kernel $k(t,x)$ by the variable t at the point $(0,x)$ under $x \in [-1,1]$. By the supposition of the theorem that the kernel $k(t,x)$ is presentable in the following way

$$k(t,x) = \sum_{k=0}^{p-1} c_k(x)\, t^k + t^p \widehat{K_1}(t,x) \quad (46)$$

where $c_k(x)$ and $\widehat{K_1}(t,x)$ are continuous functions. Using the later we find that

$$(K'\Psi)(x) = \sum_{k=0}^{p-1} c_k(x) \int_{-1}^{1} \frac{z(t)}{t^{p-k}} dt + \int_{-1}^{1} z(t)\, \widehat{K_1}(t,x)dt + \sum_{k=0}^{p-1}(-1)^j \beta_j K_1^{\{j\}}(0,x), \quad (47)$$

Here in the last formula, the first and the third terms are finite dimensional operators and, the second term is completely continuous from $C[-1,1]$ into $C[-1,1]$.

The theorem is proved.

$P_m^1[-1,1]$ to represent the function $\Psi(x)$ in the form

$$\Psi(x) = x^{-p} z(x) + \sum_{k=0}^{p-1} \beta_k\, \delta^{\{k\}}(0,x), \quad (48)$$

where

$z(x) \in C_0^{\{p\}}[-1,1]\, \mathrm{X}\, \mathrm{C}\, [-1,1]$, and $z(1) = 0$. Next, with respect of the assertion of the

Lemma compare the left and right sides of the considered relationship.

Let us finally state the main important theorem with the relation to the previous noetherisation's investigation of the integro-differential operator A.

Theorem 3. Let the integral operator (1) defined as an operator acting from $C_{-1}^{(m)}[-1,1]$ into $C_0^{\{p\}}[-1,1]$ in the supposition that the kernel $k(x,t)$ is verifying the condition $k(x,t) \in C_0^{\{p\}}[-1,1]\, \mathrm{X}\, \mathrm{C}\, [-1,1]$.

The operator $A: C_{-1}^{(m)}[-1,1] \to C_0^{\{p\}}[-1,1]$ defined by the formula (1) is noetherian operator with the index $Ind\, A = \chi(A) = -p$.

The proof of this theorem can be easily deduced from the Lemma 6 related to the noetherity of the operator L as principal part of the operator A, from the assertion of the Theorem 2 and also is based on the well-known fact that when a certain noetherian operator is perturbed by a completely continuous operator, the noetherian property of the operator and the index of the operator are preserved. On this matter it is sufficient to note that $A = L + K$.

The theorem is proven.

Next further, alongside with the operator A, defined in (1) we will consider the associated operator A' defined by the following way:

$$(A'\Psi)(x) = (-1)^m \left(x^p \Psi(x)\right)^{(m)} + \int_{-1}^{1} k(t,x)\, \Psi(t)dt = (L'\Psi)(x) + (K'\Psi)(x)\,; x \in [-1,1], \quad (49)$$

where $m > 1$, $k(t,x) \in C_0^{\{p\}}[-1,1]\, \mathrm{X}\, \mathrm{C}\, [-1,1]$.

Likewise it can be proven the following theorem.

Theorem 4. The operator $A': P_m^1[-1,1] \longrightarrow C[-1,1]$ is noetherian with the index $\chi(A') = p$.

Easily it can be established that the operators A and A' are associated operators and so that it takes place the following lemma.

Lemma 8. The operators $A: C_{-1}^{(m)}[-1,1] \to C_0^{\{p\}}[-1,1]$ and $A': P_m^1[-1,1] \longrightarrow C[-1,1]$ verify the relationship: $\int_{-1}^{1} \Psi(t)(A\varphi)(t)dt = \int_{-1}^{1} \varphi(t)\, (A'\Psi)(t)dt, \forall \varphi \in C_{-1}^{(m)}[-1,1]$ and $\forall\, \Psi \in P_m^1[-1,1]$. (50)

Poof: For the proof of the relationship (50) it is sufficient from the definition of the space $P_m^1[-1,1]$ to represent the function $\Psi(x)$ in the form

$$\Psi(x) = x^{-p} z(x) + \sum_{k=0}^{p-1} \beta_k\, \delta^{\{k\}}(0,x), \quad (51)$$

where

$z(x) \in C_0^{\{p\}}[-1,1]\, \mathrm{X}\, \mathrm{C}\, [-1,1]$, and $z(1) = 0$. Next with respect of the assertion of the Lemma compare the left and right sides of the relationship (50).

The Lemma is proved.

As A and A' are noetherian associated operators and $\chi(A) = -\chi(A')$, then we obtain the following important general theorem.

Theorem 5. The equation $A\varphi = f$, where A is the operator of the form (1) and $f(x) \in C_0^{\{p\}}[-1,1]$ is solvable in the space $C_{-1}^{(m)}[-1,1]$ if and only if they are realized the following conditions:

$\int_{-1}^{1} f(t)\, \Psi_k(t)dt = 0,\ k = 1,2,3, \ldots \alpha(A')$, where $\{\Psi_k\}-$ is the basis of the space of the solutions of the homogeneous associated equation $A'\Psi = 0$ in the associated space $P_m^1[-1,1]$.

Proof. The proof of this theorem is obtained during the whole complete process of the construction of the noetherian theory of the operator A, schematized through the preliminary investigation of the noetherization of its principal part L and of the principle of the conservation of noetherity, following its perturbation by a completely continuous integral operator K, making the initial operator A a noetherian operator having the same index as its principal part.

Let us consider an illustrative example to better understand and demonstrate the construction of the theory in question for a specific integro-differential operator.

Example. Let us consider the following integro-differential operator defined by the third kind integral equation

$$A(\varphi)(x) = x\varphi'(x) + \int_{-1}^{1} t\varphi(t)\ dt = f(x) \quad (52)$$

where $f(x) \in C_0^{\{1\}}[-1,1]$ and the solution $\varphi(x)$ is to be found from the space $C_{-1}^{1}[-1,1]$. It is not difficult to see that this equation has a solution noted

$$\varphi(x) = \int_{-1}^{1} (Nf)(t)\ dt \quad (53)$$

if and only if it is realized the condition

$$f(0) = \int_{-1}^{1} tdt \int_{-1}^{t} (Nf)(s)\ ds \quad (54)$$

After changing the order of integration and performing simple calculations, the last is written in the following form:

$$\int_{-1}^{1} [(Nf)(s) - sf(s)]\, ds - 2f(0) = 0 \quad (55)$$

or in abbreviated form

$$\left(f, \frac{1}{s} - s - 2\delta(s)\right) = 0. \quad (56)$$

It is not difficult to prove that the function of the form

$$\Psi(x) = \frac{1}{x} - x - 2\delta(x) \quad (57)$$

is the unique linear independent solution of the homogeneous associated equation

$$A'\Psi = 0 \quad (58)$$

in the associated space P^1. We recall that in this case, it is indicated to look for $\Psi(t)$ in the form of the following way

$$\Psi(t) = \frac{z(t)}{t} + \beta\delta(t), \quad (59)$$

where $z(t) \in C_0^{\{1\}}[-1,1] \cap C_1^1[-1,1]$.

Therefore, the condition (55) onto the function $f(x)$ is the condition of orthogonality to the unique non trivial solution of the associated homogeneous equation $A'\Psi = 0$.

To close our research carried out in this work, let's go straight to the next part entitled conclusion.

IV. CONCLUSION

The scientific investigations carried out in this work present in detail the different stages of the complete process of the construction of the noetherian theory for the integro-differential operator A, of which the noetherization of the operator of the principal part denoted L was firstly established, defined by a linear singular differential equation of higher order m in the space of continuous functions $C_{-1}^{(m)}[-1,1]$. Rigorous proofs were provided regarding the compactness of the integral operators, as well as the concept and principle of associated operators. Subsequently, the associated operators L and L', the two pairs of associated Banach functional spaces $C_{-1}^{(m)}[-1,1]$ with $C[-1,1]$ and $P_m^1[-1,1]$ with $C_0^{\{p\}}[-1,1]$ related to them for the construction of the noetherian theory were highlighted, and the related properties demonstrated. All the above allowed us to gradually determine the deficient and characteristic numbers (α, β) and, consequently, the indices $\varkappa(L)$ and $\varkappa(L')$ respectively of the associated operators L and L'. By applying the principle of conservation of the noetherity of a noetherian operator after its perturbation by a compact integral operator, we were able to demonstrate that the operator A as the sum of the operators L and K, where K is a compact operator defined as follows $K\varphi = \int_{-1}^{1} k(x,t)y(t)dt$ is also in turn noetherian with conservation of its index.

V. RECOMMANDATION

After having carried out the noetherization of the integro-differential operator considered in this article in the space $C_{-1}^{(m)}[-1,1]$, we can recommend, among other things, firstly, to carry out its various extensions as is done naturally after having obtained such results. In other words, it is necessary to add to the initial functional space successively by direct sum, the spaces constituted by a finite linear combination of the Dirac delta distributions as well as its successive derivatives of finite order $\left\{\sum_{j=0}^{s} \alpha_j \delta^{\{j\}}(x)\right\}$, the functional space constituted by the finite parts in the sense of Hadamard $\left\{\sum_{n=1}^{l} \alpha_n F.p\frac{1}{x^n}\right\}$, and finally the sum space of the two previous ones $\left\{\sum_{j=0}^{s} \alpha_j \delta^{\{j\}}(x)\right\} \oplus \left\{\sum_{n=1}^{l} \alpha_n F.p\frac{1}{x^n}\right\}$. This will be the subject of the contents of our next scientific investigations.

Acknowledgments

The authors are very grateful to late Professor KARAPETYANTS N.K former Head of the Department of Differential and Integral Equations of the Rostov State University in Russian Federation, for his invaluable support and helpful discussions by that time when conducting such researches.

Also we would like to thank the anonymous reviewers of this article for agreeing to read the manuscript very carefully, as well as for their valuable comments and suggestions which considerably should improve the document.

REFERENCES

[1] *(text in Russian)* Prossdorf, Some Classes of Singular Equations, Mir, Moscow, 1979. ISBN-13: 978-0-7204-0501-9.

[2] V. S. Rogozhin, Noether Theory of Operators, 2nd Edition, Rostov-na-Donu: Izdat. Rostov University, 1982, 99 pages.

[3] N. S. Karapetiants and S. G. Samko, Equations with Involutive Operators, Birkhauser, Boston–Basel–Berlin, 2001.DOI: https://doi.org/10.1007/978-1-4612-0183-0.

[4] G. R. Bart, Three theorems on third-kind linear integral equations, Journal of Mathematical Analysis and Applications 79 (1) (1981), 48-57.

[5] Shulaia D."Solution of a linear integral equation of third kind", Georgian Math J, 2002, Vol. 9, 179-196.

[6] Shulaia D. "Integral equations of third kind for the case of piecewise monotone coefficients", Transactions of A. Razmadze Math.Institute, 2017, Vol. 171, 396-410.

[7] Gobbasov N.S."Methods for Solving an Integral Equation of the Third Kind with Fixed Singularities in the Kernel", Diff. Equ, 2009, Vol. 45, 1341-1348.

[8] Sukavanam N. "A Fredholm-Type theory for third kind linear integral equations", J. Math. Analysis Appl, Vol. 100, 1984, 478-484.

[9] E. Tompé Weimbapou, Abdourahman and E. Kengne, On Delta-Extension for a Noether Operator, ISSN 1066-369X, Russian Mathematics 65(11) (2021), 34-45. c Allerton Press, Inc., 2021. Russian Text c The Author(s), 2021, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2021, No. 11, pp. 40–53. https://doi.org/10.3103/S1066369X21110050.

[10] R. V. Duduchava, Singular integral equations in the Holder spaces with weight, I. Holder coefficients, Mathematics Researches, T. V, 2nd Edition (1970), 104-124.

[11] Kravchenko, V.G., Litvinchuk, G.S. (1994). The Noether theory of a singular integral functional operator of finite order in the continuous case. In: Introduction to the Theory of Singular Integral Operators with Shift. Mathematics and Its Applications, vol 289. Springer, Dordrecht. https://doi.org/10.1007/978-94-011-1180-5_3.

[12] Rogozhin V.S, Raslambekov N.S. The Noether theory for integral equations of the third kind, Differ.Uravn., 1978, Volume 14, Number 9, 1678-1686. Math-Net.Ru All Russian Mathematical Portal.

[13] Yurko V. A. On a differential operators of higher order with singularities inside the interval. Kratkie sochenie// Mathematicheskie Zamietkie, 2002. T.71, N° 1 P152-156.

[14] Abdourahman A., Karapetiants N. "Noether theory for third kind linear integral equation with a singular linear differential operator in the main part", Proceedings of A. Razmadze Math. Institute 135, 1–26 (2004).

[15] Abdourahman Haman Adji and Shankishvili Lamara Dmitrievna. Practical illustration of the noetherization of an integro-differential operator A. International Journal of Pure Mathematics. DOI: 10.46300/91019.2025.12.1. Volume 12, 2025. E-ISSN: 2313-0571.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Abdourahman HAMAN ADJI formalized the research topic based on a theme that is part of the program of research projects of interest to researchers and systematically proposed the exhaustive methodology scheme of the investigations conducted in the whole work carried out in this draft article as well as the efficiency of all calculations.

SHANKISHVILI Lamara Dmitrievna also verified some calculations performed in the whole research conducted of the unknown coefficients of the systems of equations obtained within the framework of the whole work. Even more, it established various links in relation to the construction of the Noetherian theory of integro-differential operators in certain distributional functional spaces, specific in the objective of realizing the established extension.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

The realization of this research work partially started basically at the Chair of Differential and Integral Equations of the Faculty of Mechanics and Mathematics of the State University of Rostov-On-Don in the Russian Federation and finalized at the University of Ngaoundere in Cameroon is financed by the premium for the modernization of research from the Ministry of Higher Education of Cameroon.

We confirm that each of the authors contributed as far as they were concerned in the realization of the work carried out under the general supervision of the corresponding author, according to the distribution of the different activities.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

Creative Commons Attribution License 4.0 (Attribution 4.0 International, CC BY 4.0)

This article is published under the terms of the Creative Commons Attribution License 4.0
https://creativecommons.org/licenses/by/4.0/deed.en_US