INTERNATIONAL JOURNAL OF PURE MATHEMATICS
DOI: 10.46300/91019.2026.13.1 Volume 13, 2026

On a Right Euler Second Order Linear Singular Differential Equation in the Space K′.

Abdourahman Haman Adji[1,*], Momo Zanguim Suzie[2], Shankishvili Lamara Dmitrievna[3]

[1]Department of Mathematics and Computer Sciences, Faculty of Sciences,
The University of Ngaoundere
Cameroon

[2]Department of Mathematics,
Higher Teacher's Training College, University of Maroua
Cameroon

[3]Department of Mathematics, Georgian Technical University,
Kostova Street, Tbilisi,
Georgia

*Corresponding author

Received: March 6, 2026. Revised: June 4, 2026. Accepted: August 8, 2026. Published: September 23, 2026.

Abstract— In this article, we study the solvability of a singular second-order linear differential equation in the distributional space K'. We determine all the distributional solutions centered at zero of the equation under consideration. Several of our previous studies, whose results have been widely published in mathematical journals, address with this theme in various ways depending on the types of equations considered. To achieve the set goal, the unknown function $y(x)=\sum_{k=0}^{N} C_k\,\delta^{(k)}(x)$ is a distribution, presented as a functional and taken in the form of a finite linear combination of Dirac distributions and their ordinary derivatives. The methodology consists of substituting it into the original equation, which leads to the laborious solution of a system of equations to determine the undetermined coefficients C_k and, once found, to give the form of the general distributional solution centered at zero.

Keywords—Test functions, Generalized functions, Distributional function, Dirac Delta distribution, Fourier Transform, distributional zero-Centered Solutions.

Mathematics Subject Classification: 34A05, 39A05, 42B10, 44A05.

I. INTRODUCTION

Historically speaking, differential equations appeared at the very beginning of the development of classical mathematical analysis, generally during the search for solutions to problems in mechanics or geometry.

Ordinary linear differential equations, widely used in different branches of science and in the study of physical phenomena,

not only allow us, through their solutions, to better understand and interpret the study carried out, but also stimulate further research and new scientific questions.

The encyclopedia on differential equations lists a large number of types of linear differential equations as well as the recommended methods for solving them. However, if in the first investigations carried out, one endeavored specially to calculate the solutions of them by means of functions already known, then it turned out without delay that this point of view was too narrow. In reality, the fundamental problem of the theory of differential equations is that of deducing the properties of the solutions of a given equation or system from the analytical form of these. All the same, in general, we note that the equations which result from a theoretical investigation in mathematics or physics are not explicitly integrable and constitute, very often, the main source for the definition of new special functions whose properties can be predicted by a systematic analysis of large classes of equations or differential systems. Solving a differential equation remains a challenge when the type of it goes beyond the canvas of the standards of what we know how to do classically.

Several scientists have published results of their work related to the search for solutions to certain differential equations of times in specific functional spaces. Also, we could rightly cite the scientific works and investigations relating to it to a certain extent to the problem of the solvability of the different types of singular linear differential equations, undertaken in certain distributional spaces based on the theories presented in books and scientific results papers set out in the references, [1], [2],

[3], [4], [5], [6], [7], [8], [9].

In the same vein and continuing in the same direction, we can mention our previous scientific results carried out and published in the scientific journal indicated by [10], based on the rich scientific theories presented in the very well-known published references, [11], [12], [13], [14], [15]. In our previously published results, we have completely completed the research work of investigating the first-order equation of type:

$Ax^p y'(x) + Bx^q y(x) = \delta^{(s)}(x)$ (1),

when we have analytically described all the distributional zero-centered solutions and classical solutions in the space K' according to the different ratios between the parameters A, B, p, q and s. Interesting results have been obtained when we conducted investigations of such a previous equation in a larger distributional space, noted $\left(S_0^{\beta}\right)'$ than the space K'.

Naturally, we are led to set ourselves the general objective of achieving the investigation of the situation of a general equation of order l, that is to say, to try to reach the general situation when investigating the singular differential equation defined as follows:

$\sum_{i=0}^{l} a_i\, x^{k_i} y^{(i)}(x) = \delta^{(s)}(x)$ (2),

with $k_i = k_{i-1} + 1;\ \ i = 1\ldots\ldots, l-1$

but $k_l \neq k_{l-1}$ +1, i.e., $k_l > k_{l-1}$ +1 or $k_l < k_{l-1} + 1$.

Among others, the objective of carrying out these investigations related to the search for zero-centered distributional solutions is also connected to the project of the construction of the Noetherian theory of integro-differential operator A defined by the equation:

$Ay(x) = Ly(x) + Ky(x) = f(x)$ (3),

as a singular linear integral equation of the third kind, containing in its main part the differential operator L, where

$Ly(x) = Ax^m y''(x) + Bx^n y'(x) + Cx^r y(x)$ (4).

In the investigated cases of the finite- dimensional extensions of the integro-differential operators A with elements from the space $D_m = C_{-1}^{1}[-1,1] \oplus \{\sum_{k=0}^{m} \alpha_k \delta^{\{k\}}(x)\};\ m > p-2$, we studied the noetherization of the extended initial noetherian operator noted $\bar{A}$ when we realize the extension of the unknown function $\varphi(x)$ from the space D_m. The realization of the necessary extension also leads us to the situation of the investigation of the differential equation of such evoked forms, hence also the interest of carrying out this research, which goes directly in line with our work in the field of the construction of the Noetherian theory.

Our present attention is focused on the continuity of the spirit of analysis and resolution of a type of second- order linear singular differential equation called the right Euler type equation in the space of distributions K'.

Namely, in this work, we carry out the study of the solvency of the linear singular differential equation of second order in the space K' and describe all the zero-centered distributional solutions. In other words, we are looking for all the zero-centered distributional solutions to the said equation, defined by the following formula (5) in K' the space of distributions indicated,

$Ax^m y''(x) + Bx^n y'(x) + Cx^r y(x) = \delta^{(s)}(x)$ (5)

where s is a natural integer, and the coefficients A, B, and C are real numbers not all zero, $m, n, r, s \in \mathbb{N}$, $\delta^{(0)}(x) = \delta(x)$ is the Dirac delta distribution centered at 0, and $\delta^{(s)}(x)$, is the s^{th} derivative with respect to, finally $m = n+1,\ r < n-1$.

Recall that in our previous publications, the search for zero-centered distributional solutions of the same type of differential equation had been carried out in all its details in the case where the equation was of Euler type when $m = n+1 = r+2$, and of right Euler type when $m = n+1,\ r > n-1$.

The work is organized as follows. First, we present in Section II the preliminaries related to the concept and the notions of the well-known distributional theory. Section III is properly devoted to the presentation of the main results of this work and is devoted to the non-degenerate right Euler case studied, with the parameters A, B, and C, such that their product $A \cdot B \cdot C \neq 0$ and $m = n+1,\ r < n-1$. And lastly, we summarize the content of the work in section IV titled conclusion, followed by some recommendations for the follow-up or future scientific works to undertake, stated in section V.

II. Preliminaries

For the execution of these investigations, we will make a brief reminder of some important notions relating to the notion of the Fourier transform, its essential properties, and a distribution centered at a given point. Further details on these concepts may be found easily in various monographs and scientific books containing the different theories on test functions and distributions in several distributional spaces.

It should be remembered that K denotes the space of test functions, that is to say, indefinitely differentiable and identically zero functions outside a set bounded on $\mathbb{R}^1$, while K' is noted as the space of distributions or generalized functions on K. We recall that the sth-order derivative of a distribution T, denoted by $T^{(s)}$, is defined by:

$\left(T^{(s)}, \varphi(x)\right) = (-1)^s \left(T, \varphi(x)^{(s)}\right)$ for all $\varphi(x) \in K$ (6).

Let give an example of derivatives of the singular distribution

T=δ we have:

$$\begin{cases} a) \quad \left(\delta'(x), \varphi(x)\right) = -\left(\delta(x), \varphi'(x)\right) = -\varphi'(0) \\ b) \quad \left(\delta^{(s)}, \varphi(x)\right) = (-1)^s \left(\delta(x), \varphi(x)^{(s)}\right) = (-1)^s \varphi(0)^{(s)} \end{cases}$$
(7).

For the function $\varphi(t) \in K$, through $F\varphi = \hat{\varphi}$, we designate the Fourier transform defined by the relationship:

$(F\varphi)(x) = \hat{\varphi}(x) = \int_{-\infty}^{+\infty} \varphi(t) e^{ixt} dt$

(8)

We define the Fourier transform of the distribution (generalized function) $f \in K'$ we define by the rule (Parseval equality):

$\left(\hat{f}, \hat{\varphi}\right) = 2\pi (f, \varphi)$ (9)

We can note that for the Fourier transform of a distribution (generalized function), there is conservation of many known

properties, like those noted and relating to the Fourier transform of the test functions, in particular, the relation between the differentiability and the average decrease of the formulas. It follows in particular that:
$F\left[\delta^{(s)}(t)\right] = (-ix)^s,\ s \in \mathbb{N} \cup \{0\}.$
(10)

We will use the following statements, which can be found in standard references on distribution theory.

Theorem 1. If $f, g \in K'$ and $f' = g'$ then $f - g = c$.

Theorem 2. Let $A(x) \in C^\infty(\mathbb{R}^1)$. The linear differential equation $y' = A(x)y$ in the space K' does not admit other solutions that are not classical solutions.

Definition 1. A distributional function (generalized function) $f \in K'$ is called centered at the point x_0, if $\left(f, \varphi(x)\right) = 0$ for all $\varphi(x) \in K$ such $x_0 \in Supp\varphi$.

Theorem 3. Let $f \in K'$ centered at zero. Then there exists $m \in \mathbb{N} \cup \{0\}$ such that:

$$f(x) = \sum_{j=0}^{m} c_j \delta^{(j)}(x) \tag{11}$$

where c_j-are some constants.

Lemma 1. Let $\beta(x) \in C^\infty(\mathbb{R}^1)$. Then it holds the following formula:
$\beta(x)\delta^{(j)}(x) =$
$\sum_{j=0}^{l}(-1)^j C_l^j \beta^i(0)\, \delta^{(l-j)}(x).$
(12)

The proof of Lemma 1 is given in [9].

As a consequence, following Lemma 1,

when we set $\beta(x) = x^k$, we obtain the next assertion.

Lemma 2. Let $s \in \mathbb{N} \cup \{0\}$. Then

$$x^k\delta^{(s)}(x) = \begin{cases} 0, & if\ s < k \\ \frac{(-1)^k s!}{(s-k)!}\delta^{(s-k)}(x), & if\ s \geq k. \end{cases}$$
(13)

As a remark, implicitly in the analysis of the different cases of the degenerate situation of equation (5), we arrive at the investigation of the following expressions: $\frac{\delta^{(s)}(x)}{x^m}$, $\frac{\delta^{(s)}(x)}{x^n}$ and $\frac{\delta^{(s)}(x)}{x^r}$. This being so, we will understand one of these expressions in the sense of the following definition.

Definition 2. The quotient $\frac{\delta^{(s)}(x)}{x^m}$ is called a distributional function designated by $y(x) \in K'$, which satisfies in the space K' the equality $x^m y(x) = \delta^{(s)}(x)$ i:e

$$\left(x^m y(x), \varphi(x)\right) = \left(\delta^{(s)}(x), \varphi(x)\right) = (-1)^s \varphi^{(s)}(0),$$
$\forall \varphi \in K.$ (14).

Now we can easily deduce from Lemma 2 and Definition 2 the following formula for the exact computation of the analytic expression of the generalized function defined by the expression $\frac{\delta^{(s)}(x)}{x^m}$.

Let us formulate the result of this calculation in the form of the following Lemma.

Lemma 3. Let $m \in \mathbb{N}, s \in \mathbb{N} \cup \{0\}$. It holds the following formula.

$$\frac{\delta^{(s)}(x)}{x^m} = \frac{(-1)^m s!\delta^{(m+s)}(x)}{(s+m)!} + \sum_{k=1}^{m} c_k \delta^{(k-1)}(x)$$

where C_k, k = 1, ...m - are arbitrary constants. (15)

Proof.

Let us use definition 2 and apply the Fourier transform to both sides of the equality $x^m y(x) = \delta^{(s)}(x)$, with consideration that $\widehat{(\imath x)^m y}(x) = \widehat{y(x)}(\varepsilon)$.

Finally, we have

$\hat{y}^m(\varepsilon) = \widehat{(\imath x)^m y}(x) = i^m \widehat{\delta^{(s)}(x)} = i^m(-i\varepsilon)^s$.

From the previous, we reach

$\hat{y}(\varepsilon) = P_{m-1}(\varepsilon) + \frac{(-1)^m s!}{(s+m)!}(-i\varepsilon)^{s+m}$ where $P_{m-1}(\varepsilon)$ is a polynomial with arbitrary coefficients. Applying the inverse Fourier transform with respect to the relationship (10), we reach the needed result and arrive at the formula (15). So, consequently, the lemma is proved.

Remark 1. We note that in the analytical expression of the result of the "quotient or ratio" of the Dirac delta distribution $\delta^{(s)}(x)$ by x^m due to the singularity "union", not only does the order of the derivative of the Dirac delta distribution increase, but there is the appearance of an arbitrary dependence on m.

Now, let's get straight to the most important part of our presentation in a comprehensive manner.

III. MAIN GLOBAL RESULTS.

A. *Right Euler Situation. The non-degenerate case.*

We focus our investigation on the situation called the right Euler case when the following conditions are realized:

$$A \cdot B \cdot C \neq 0 \text{ and } m = n + 1, \quad r < n - 1 \tag{16}$$

for the non $-$ homogeneous equation.

This research aims to investigate the general situation when investigating the linear singular differential equation defined by (2).

Theorem3. Let the product $A.B.C \neq 0, m, n \in \mathbb{N}$, $r, s \in \mathbb{N} \cup \{0\}$ and accomplish the condition (16), then the homogeneous equation (5) admits a distributional zero-centered solution of the following form:
$y(x) =$
$\sum_{j=0}^{r-1} c_j \delta^{(j)}(x),$ (17)

where $c_j, j = 0, \ldots, r-1$ – arbitrary constants.

Proof. The proof can be deduced from a simple analysis of the determining system for finding coefficients and the application of Lemma 2.

B. *Non-homogeneous equation.*

Let us consider the non-homogeneous equation (5) and find the distributional zero-centered solutions. For this matter, we should look for the particular solutions in the form of a functional centered at the point zero as defined by the formula:

INTERNATIONAL JOURNAL OF PURE MATHEMATICS
DOI: 10.46300/91019.2026.13.1 Volume 13, 2026

$y(x) = \sum_{j=0}^{\gamma_0} C_j \, \delta^{(k)}(x)$ (18)

where $C_j -$ unknown coefficients are to be determined.

Simple computations when putting (18) into (5) and with respect to Lemma 1 lead us to the following result.

$$\mathrm{A}x^{n+1}\sum_{j=0}^{\gamma_0} c_j\delta^{(j+2)}(x) + \mathrm{B}x^{n}\sum_{j=0}^{\gamma_0} c_j\delta^{(j+1)}(x)$$
$$+\mathrm{C}x^{r}\sum_{j=0}^{\gamma_0} c_j\delta^{(j)}(x) = \sum_{s=0}^{\mu}\delta^{(s)}(x) \quad (19)$$

or with the dependence of the relationship between the parameters r, and $n-1$ we obtain two possibilities:

Let $r < n-1$, then (19) has the following form:

$$\sum_{k=0}^{\gamma_0+1-n}\left[c_{k+n-1}(-1)^n\frac{(k+n)!}{k!}[B - A(k+n+1)] + C(-1)^r c_{k+r}\frac{(k+r)!}{k!}\right]\cdot\delta^{(k)}(x) + C\sum_{k=\gamma_0+2-n}^{\gamma_0-r} c_{k+r}(-1)^n\frac{(k+r)!}{k!}\delta^{(k)}(x) = \delta^{(s)}(x) \quad (20).$$

Next, in this part, we look for a particular solution when $r < n-1$.

As we already obtained the solution of the homogeneous equation in the form (17) from Theorem 3.

When searching for a particular solution of equation (5), we arrive at system (20). The analysis of this system shows that it is necessary to differentiate between the following two situations:

$a)$ $s+r < n-1$ and $b)$ $s+r \geq n-1$.

Let $s+r < n-1$ and put the particular solution as

$y_p(x) = R_0\delta^{(s+r)}(x),$ (21)

where R_0 it is not yet defined. In this case, setting the expression for $y_p(x)$ in (5), we obtain:

$$R_0\left(Ax^{n+1}\delta^{(s+r+2)}(x) + Bx^n\delta^{(s+r+1)}(x) + Cx^r\delta^{(s+r)}(x)\right) = \delta^{(s)}(x). \quad (22)$$

As $s+r < n-1$, then

$(s+r+2)-(n+1) = s+r-n+1 < 0,$

$(s+r+1)-n = s+r-n+1 < 0,$

and, consequently, $x^{n+1}\delta^{(s+r+2)}(x) = 0,$

$x^n\delta^{(s+r+1)}(x) = 0$

by virtue of the application of Lemma 2.

Consequently, from (22) we obtain:

$R_0 \; C(-1)^r\frac{(s+r)!}{s!} = 1,$ (23)

From which we immediately obtain.

$R_0 = (-1)^r\frac{s!}{C(s+r)!}$ (24)

Therefore $y_p(x) = (-1)^r\frac{s!}{C(s+r)!}\delta^{(s+r)}(x).$ (25)

$b)$ In the case $s+r \geq n-1$, following our general ideas from previous research published, we should look for the particular in the following way:

$y_p(x) = \sum_{l=1}^{l_0} R_l\delta^{(s+r-l(n-1-r))}(x) + R_0 \; \delta^{(s+r)}(x)$ (26)

where l_0 is defined by the conditions

$r \leq s+r-l_0(n-1-r) < n-1.$ (27)

The latter means that:

$$\left.\begin{matrix} l_0 \leq \frac{s}{n-1-r} \\ l_0+1 > \frac{s}{n-1-r}\end{matrix}\right| \Rightarrow l_0 = \left[\frac{s}{n-1-r}\right] \quad (28)$$

Let us consider, at the beginning, the particular case which will indicate to us how to manage the general case. Let $l_0 = 1$, and moreover

$s+r = n-1$ (29)

In this case, in (22), we get two terms and

$y_p(x) = R_1\delta^{(r)}(x) + R_0 \; \delta^{(s+r)}(x)$ (30)

Note that $s \neq 0$ with respect by virtue to $r \neq n-1$, so that $s > 0$. After substitution (30) into (1), we have:

$$R_1\left[Ax^{n+1}\delta^{(r+2)}(x) + Bx^n\delta^{(r+1)}(x) + Cx^r\delta^{(r)}(x)\right] + R_0\left[\begin{matrix}Ax^{n+1}\delta^{(s+r+2)}(x) + Bx^n\delta^{(s+r+1)}(x) + \\ Cx^r\delta^{(s+r)}(x)\end{matrix}\right] = \delta^{(s)}(x).$$
(31)

Considering that

$(r+2)-(n+1) = r-(n-1) = -s < 0$

i.e $r+1-n < 0$, we obtain at the end:

$$R_1(-1)^r Cr!\,\delta(x) + R_0[A(-1)^{n+1}(s+r+2)!\,\delta(x) + B(-1)^n(s+r+1)!\,\delta(x)] + R_0C(-1)^r\frac{(s+r)!}{s!}\delta^{(s)}(x) = \delta^{(s)}(x). \quad (32)$$

To find the coefficients R_0, R_1, we reach the following system:

$$\begin{cases} R_1(-1)^r Cr! + R_0(-1)^n(s+r+1)! \\ \quad\cdot[B - A(s+r+2)] = 0 \\ \quad R_0C(-1)^r\frac{(s+r)!}{s!} = 1. \end{cases} \quad (33)$$

From the previous, it follows that:

$R_0 = (-1)^r\frac{s!}{C(s+r)!}$ (34)

$$\mathrm{R}_1 = \frac{(-1)^{n-r+1}(s+r+1)!}{\mathrm{C}\,r!}\cdot[\mathrm{B}-\mathrm{A}(s+r+2)]\mathrm{R}_0 \quad (35)$$

and if we make the substitution of these coefficients into (30), then we obtain the following result for the particular solution $y_p(x)$.

$$y_p(x) = \frac{(-1)^r\,s!}{C(s+r)!}\delta^{(s+r)}(x) + \frac{(-1)^{n-1}}{C^2}\frac{(s+r+1)!\,s!}{r!\,(s+r)!} \times [B - A(s+r+2)]\delta^{(r)}(x). \quad (36)$$

Let's move on to the general case. By the definition of the number l_0

$s+r-l_0(n-1-r)+2-(n+1) = s-(l_0+1)(n-1-r) < 0.$

Therefore, two terms disappear in the sum below. Namely,

$$\sum_{l=1}^{l_0-1} R_l\left[\begin{matrix} A\dfrac{(-1)^{n+1}[s+r-l(n-1-r)+2]!}{[s-(l+1)(n-1-r)]!} + \\ B\dfrac{(-1)^n[s+r-l(n-1-r)+1]!}{[s-(l+1)(n-1-r)]!}\end{matrix}\right]\cdot\delta^{(s-(l+1)(n-1)r)}(x)$$
$$+\sum_{l=1}^{l_0} R_l\,(-1)^r C\frac{[(s+r)-l(n-1-r)]!}{[s-l(n-1-r)]!}\delta^{(s-l(n-1-r))}(x)$$

$$\cdot R_0\left[\frac{A(-1)^{n+1}[s+r+2]!}{[(s+r+2)-(n+1)]!}+\frac{B(-1)^{n}(s+r+1)!}{(s+r-n+1)!}\right]\delta^{(s+r-n+1)}(x)+R_0(-1)^r\frac{(s+r)!}{s!}\delta^{(s)}(x)=\delta^{(s)}(x). \quad (37)$$

Next, we make the substitution in the first sum $l+1\longrightarrow l$. Then, from the previous, it follows:

$$\sum_{l=2}^{l_0} R_{l-1}\left[A\frac{(-1)^{n+1}[s+r-(l-1)(n-1-r)+2]!}{[s-l(n-1-r)]!}+B\frac{(-1)^{n}[s+r-(l-1)(n-1-r)+1]!}{[s-l(n-1-r)]!}\right]\delta^{(s-l(n-1-r))}(x)+\sum_{l=1}^{l_0} R_l\,(-1)^r C\frac{[s+r-l(n-1-r)]!}{[s-l(n-1-r)]!}\delta^{(s-l(n-1-r))}(x)+R_0\left[\frac{A(-1)^{n+1}(s+r+2)!}{(s+r-n+1)!}+\frac{B(-1)^{n}(s+r+1)!}{[s+r-(n-1)]!}\right]\delta^{(s+r-n+1)}(x)+R_0(-1)^r\frac{(s+r)!}{s!}\delta^{(s)}(x)=\delta^{(s)}(x). \quad (38).$$

It is not difficult to see that the term under $\delta^{(s+r-n+1)}(x)$ can be included in the first sum.

Equalizing the coefficients under the delta function and its derivatives, we obtain the following system:

$$\begin{gathered} R_0C(-1)^r\frac{(s+r)!}{s!}=1\\ R_lC(-1)^r[s+r-l(n-1-r)]!\\ +R_{l-1}(-1)^n\cdot\\ [s+r-(l-1)(n-1-r)+1]!\cdot\\ [B-A(s+r-(l-1)(n-1-r)+2)]=0. \end{gathered} \quad (39)$$

From which we obtain:

$$R_0=\frac{(-1)^r s!}{C(s+r)!}, \quad (40)$$

$$R_l=\frac{(-1)^{n-r+1}[s+r-(l-1)(n-1-r)+1]!}{C[s+r-l(n-1-r)]!}\times[B-A(s+r-(l-1)(n-1-r)+2)]R_{l-1}, \quad (41)$$

From here, we find successively R_l.

But before, let us remark that when $(l_0=1)$ then and $l=1$ we have:

$$R_1=\frac{(-1)^{n-r+1}(s+r+1)!}{C[s+r-(n-1-r)]!}[B-A(s+r+2)]R_0, \quad (42)$$

What coincides with (35) when $s+r=n-1$. From (41) we have the following result:

$$R_l=(-1)^{(n-r+1)l}C^{-l}\prod_{j=1}^{l}\frac{[s+r-(j-1)(n-1-r)+1]!}{[s+r-j(n-1-r)]!}\times[B-A(s+r)+(j-1)(n-1-r+2)]. \quad (43)$$

Indeed, we have:

$$\frac{R_l}{R_{l-1}}=\frac{(-1)^{(n-1-r)l}C^{-l}}{(-1)^{(n-1-r)(l-1)}C^{-l+1}}\times\frac{\prod_{j=1}^{l}\dfrac{[s+r-(j-1)(n-1-r)+1]!}{[s+r-j(n-1-r)]!}}{\prod_{j=1}^{l-1}\dfrac{[s+r-(j-1)(n-1-r)+1]!}{[s+r-j(n-1-r)]!}}=\frac{(-1)^{n-1-r}}{C}\frac{[[(s+r)-(l-1)(n-1-r)+1]!]}{[(s+r)-l(n-1-r)+1]!}\times[B-A(s+r)+(l-1)(n-1-r)+2] \quad (44)$$

So,

$$y_p(x)=\frac{(-1)^r s!}{C(s+r)!}\delta^{(s+r)}(x)+\sum_{l=1}^{l_0}\begin{pmatrix}(-1)^{(n-1-r)l}C^{-l}\prod_{j=1}^{l}\frac{[s+r-(j-1)(n-1-r)+1]!}{[s+r-j(n-1-r)]!}\times\\ [B-A(s+r)+(j-1)(n-1-r)+2]\end{pmatrix}\times\delta^{(s-l(n-1-r))}(x). \quad (45)$$

Note that in the case when $s+r<n-1$, we can include in (45), referring to $l_0=0$ and setting $\sum_1^0 \quad =0$.

Remark 2. We can easily see that the non-homogeneous differential equation is solvable if and only if

$$B-A\big(s+n+1-j(r-n+1)\big)\neq 0, j=0,1,\ldots\ldots,l_0. \quad (46)$$

Now, having collected all these considerations, we can finally write down the general distributional zero-centered solutions of the non-homogeneous equation (5) under accomplishment of the condition $A.B.C\neq 0$; $m=n+1$, $r<n-1$.

Therefore, let us formulate the following theorem.

Theorem4. Let the product $A.B.C\neq 0, m,n\in\mathbb{N}$, $r,s\in\mathbb{N}\cup\{0\}$ accomplish the following conditions $m=n+1$, $r<n-1$,

$B-A\big(s+n+1-j(r-n+1)\big)\neq 0, j=0,1,\ldots\ldots,l_0$.

The general zero-centered solutions of the non-homogeneous equation (1) are defined by the following form:

$a)\ s+r<n-1$

$$y(x)=\sum_{j=0}^{r-1}c_j\delta^{(j)}(x)+\frac{(-1)^r s!}{C(s+r)!}\delta^{(s+r)}(x), \quad (47)$$

where $c_j, j=0,\ldots,r-1$- are arbitrary constants.

$b)\ s+r=n-1$

$$y(x)=\sum_{j=0}^{r-1}c_j\delta^{(j)}(x)+\frac{(-1)^r s!}{C(s+r)!}\delta^{(s+r)}(x)+\frac{(-1)^{n-1}}{C^2}\frac{(s+n+1)!s!}{r!(s+r)!}[B-A(s+n+2)]\delta^{(r)}(x) \quad (48)$$

where $c_j, j=0,\ldots,r-1$- are arbitrary constants.

$c)\ s+r\geq n-1,$

$$y(x)=\sum_{j=0}^{r-1}c_j\delta^{(j)}(x)+\frac{(-1)^r s!}{C(s+r)!}\delta^{(s+r)}(x)+\sum_{l=1}^{l_0}\begin{pmatrix}(-1)^{(n-1-r)l}C^{-l}\prod_{j=1}^{l}\frac{[s+r-(j-1)(n-1-r)+1]!}{[s+r-j(n-1-r)]!}\times\\ [B-A(s+r+(j-1)(n-1-r)+2)]\end{pmatrix}\times\delta^{(s-l(n-1-r))}(x) \quad (49).$$

where c_j, $j=0,\ldots,r-1$- are arbitrary constants.

Proof: the proof of Theorem 4 is obtained in the course of the construction of the solutions.

Remark: As we can easily see, the analytical expressions defining the general distributional solutions centered at zero of the studied equation are quite voluminous, and quite naturally, it is still a challenge to verify that these formulas actually verify and satisfy the identity of the term of the left-hand side to that of the right-hand side.

Before concluding, let us explain the fact that the distributional function $y(x) \in K'$ is a solution in the sense of the distributions of equation (1) if and only if:

$$\forall \varphi(x) \in K$$

$$\left(Ax^m y''(x) + Bx^n y'(x) + Cx^r y(x), \varphi(x)\right) = \left(\delta^{(s)}(x), \varphi(x)\right) = (-1)^s \varphi^{(s)}(0). \qquad (50).$$

For example, in cases where the conditions are accomplished:

$$b)\; s + r = n - 1$$

We should compute and verify that it holds the following relationship.

$$\Bigg(Ax^m\Bigg(\sum_{j=0}^{r-1} c_j\delta^{(j)}(x) + \frac{(-1)^r s!}{C(s+r)!}\delta^{(s+r)}(x) + \frac{(-1)^{n-1}}{C^2}\frac{(s+n+1)!s!}{r!(s+r)!}[B - A(s+n+2)]\delta^{(r)}(x)\Bigg)'' +$$
$$Bx^n\Bigg(\sum_{j=0}^{r-1} c_j\delta^{(j)}(x) + \frac{(-1)^r s!}{C(s+r)!}\delta^{(s+r)}(x) + \frac{(-1)^{n-1}}{C^2}\frac{(s+n+1)!s!}{r!(s+r)!}[B - A(s+n+2)]\delta^{(r)}(x)\Bigg)' +$$
$$Cx^r\Bigg(\sum_{j=0}^{r-1} c_j\delta^{(j)}(x) + \frac{(-1)^r s!}{C(s+r)!}\delta^{(s+r)}(x) + \frac{(-1)^{n-1}}{C^2}\frac{(s+n+1)!s!}{r!(s+r)!}[B - A(s+n+2)]\delta^{(r)}(x)\Bigg), \varphi(x)\Bigg) =$$
$$\left(\delta^{(s)}(x), \varphi(x)\right) = (-1)^s \varphi^{(s)}(0). \qquad (51).$$

$c)\; s + r \geq n - 1$. As in the previous part, we should meticulously realize the computations and verifications to establish that it holds the following, establishing the truth of the next equality.

$$\Bigg(Ax^m\Bigg(\sum_{j=0}^{r-1} c_j\delta^{(j)}(x) + \frac{(-1)^r s!}{C(s+r)!}\delta^{(s+r)}(x) + \sum_{l=1}^{l_0}\left(\begin{array}{c}(-1)^{(n-1-r)l}C^{-l}\prod_{j=1}^{l}\frac{[s+r-(j-1)(n-1-r)+1]!}{[s+r-j(n-1-r)]!}\times \\ [B - A(s+r+(j-1)(n-1-r)+2)]\end{array}\right)\times \delta^{(s-l(n-1-r))}(x)\Bigg)'' +$$
$$Bx^n\Bigg(\sum_{j=0}^{r-1} c_j\delta^{(j)}(x) + \frac{(-1)^r s!}{C(s+r)!}\delta^{(s+r)}(x) + \sum_{l=1}^{l_0}\left(\begin{array}{c}(-1)^{(n-1-r)l}C^{-l}\prod_{j=1}^{l}\frac{[s+r-(j-1)(n-1-r)+1]!}{[s+r-j(n-1-r)]!}\times \\ [B - A(s+r+(j-1)(n-1-r)+2)]\end{array}\right)\times \delta^{(s-l(n-1-r))}(x)\Bigg)' +$$
$$Cx^r\Bigg(\sum_{j=0}^{r-1} c_j\delta^{(j)}(x) + \frac{(-1)^r s!}{C(s+r)!}\delta^{(s+r)}(x) + \sum_{l=1}^{l_0}\left(\begin{array}{c}(-1)^{(n-1-r)l}C^{-l}\prod_{j=1}^{l}\frac{[s+r-(j-1)(n-1-r)+1]!}{[s+r-j(n-1-r)]!}\times \\ [B - A(s+r+(j-1)(n-1-r)+2)]\end{array}\right)\times \delta^{(s-l(n-1-r))}(x)\Bigg), \varphi(x)\Bigg) = (-1)^s \varphi^{(s)}(0). \qquad (52).$$

IV. CONCLUSION.

In the investigations carried out in this article, the set of all distributional solutions centered at zero of the second-order linear singular differential equation in the space of distributions K' is completely described, the right-hand side of which is the s-order derivative of the Dirac delta distribution. The methodology for searching for analytical solutions is based on the strict analysis of the system of equations, allowing the definition of the undetermined coefficients according to the different relations between the parameters of this equation. An exhaustive inventory of all solutions on a case-by-case basis was carried out, and the results obtained are the subject of the content of the theorems formulated in this article. The end of this work is devoted to the outline of the approach for verifying the general distributional solution of the equation considered in a particular case of the ratio of the parameters appearing therein.

V. RECOMMENDATION

A priori, the natural intuition arising after having searched for all the distributional solutions centered at zero of equation (5) is that of generalizing the investigations, by considering an equation of the same type, but this time of general order l, that is to say of the following form:

$$Ax^m y^{(l)}(x) + Bx^n y^{(l-1)}(x) + Cx^r y^{(l-2)}(x) = \delta^{(s)}(x), \quad l \geq 2. \qquad (53).$$

In such a project, it will be necessary to bring into play the recipes on the integration of the said distributional solutions of the present situation, studied in this article, with $l = 2$ after having posed the suitable change of variable

$$u(x) = y^{(l-2)}(x) \quad (54),$$

to return to the standard situation studied $Ax^m u''(x) + Bx^n u'(x) + Cx^r u(x) = \delta^{(s)}(x)$ before moving on to the integration of the distributional $u(x)$.

This will necessarily be the subject of the content of our future projects, which will go as far as the application of the principle of superposition of solutions during the investigation of the general order equation of the following form

$$Ax^m y^{(l)}(x) + Bx^n y^{(l-1)}(x) + Cx^r y^{(l-2)}(x) = \sum_{s=0}^{w} \delta^{(s)}(x), \; l \geq 2 \qquad (55).$$

ACKNOWLEDGMENTS

We would like to thank the anonymous reviewers for their valuable comments and suggestions on this article for agreeing to read the manuscript carefully, as well as for their valuable comments and suggestions, which considerably improved the document.

References

[1]. Sato, M. (1958). On a generalization of the concept of functions. *Proceedings of the Japan Academy*, *34*(3), 126-130. https://doi.org/10.3792/pja/1195524746.
[2]. Mirumbe, G. I., & Mango, J. M. (2018). On generalized solutions of locally Fuchsian ordinary differential equations, 51;99-117, 2018. http://dx.doi.org/10.18642/jmsaa_7100121959 2010.
[3]. Sato, M., Kawai, T., & Kashiwara, M. (1972). On the structure of single linear pseudo-differential equations. *Proceedings of the Japan Academy*, *48*(9), 643-646. https://doi.org/10.3792/pja/1195519535.
[4]. Chanapun, P., & Kaewnimit, K. (2023). The Generalized Solutions of Second and Third-Order Cauchy-Euler Equations by Using the Elzaki Transforms. *Burapha Science Journal*, *28*(1 January-April), 20-35. https://li05.tci-thaijo.org/index.php/buuscij/article/view/1186. Accessed: July 5, 2026.
[5]. Sangsuwan, A., Nonlaopon, K., & Orankitjaroen, S. (2020). The generalized solutions of a certain n th order Cauchy–Euler equation. *Asian-European Journal of Mathematics*, *13*(02), 2050047. https://doi.org/10.1142/S1793557120500473.
[6]. Liangprom, A., & Nonlaopon, K. (2017). On the generalized solutions of a certain fourth order Euler equations. *J. Nonlinear Sci. Appl*, *10(8)*, 4077-4084. https://doi.org/10.22436/jnsa.010.08.04.
[7]. Jhanthanam, S., Nonlaopon, K., & Orankitjaroen, S. (2019). Generalized solutions of the third-order Cauchy-Euler equation in the space of right-sided distributions via Laplace transform. *Mathematics*, *7*(4), 376. https://doi.org/10.3390/math7040376.
[8]. Adji, A. H., & Dmitrievna, S. L. (2023). On Classical and Distributional Solutions of a Higher Order Singular Linear Differential Equation in the Space K'. *International Journal of Pure Mathematics*, *10*, 1-7. https://doi.org/10.18642/jmsaa_7100122244.
[9]. Opio, I., Mirumbe, G. I., Ssebuliba, J., & Mango, J. M. (2017). On the solution space of ordinary differential equations with polynomial coefficients. *Far East Journal of Mathematical Sciences*, *101*(1), 103. http://dx.doi.org/10.17654/MS101010103.
[10]. Kananthai, A. (2001). The Distribution Solutions of Ordinary Differential Equation with Polynomial Coefficients: A. Kananthai. *Southeast Asian Bulletin of Mathematics*, *25*(1), 129-134. https://doi.org/10.1007/s10012-001-0129-7.
[11]. Wiener, J., & Cooke, K. L. (1990). Coexistence of analytic and distributional solutions for linear differential equations, I. *Journal of Mathematical Analysis and Applications*, *148*(2), 390-421. https://doi.org/10.1016/0022-247X(90)90009-5 .
[12]. Cooke, K. L. & Wiener, J. (1990). Coexistence of analytic and distributional solutions for linear differential equations, I. *Journal of Mathematical Analysis and Applications*, *148*(2), 390-421. http://dx.doi.org/10.1155/S0161171283000216.
[13]. Littlejohn, L. L., & Kanwal, R. P. (1987). Distributional solutions of the hypergeometric differential equation. *Journal of mathematical analysis and applications*, *122*(2), 325-345. https://doi.org/10.1016/0022-247X(87)90263-0.
[14]. Brézis, H. (2022). Linear differential operators and boundary value problems revisited. *Advances in Differential Equations*, 27, 2022. https://projecteuclid.org/journals/advances-in-differential-equations Accessed June 6, 2026.
[15]. Hörmann, G., & Spreitzer, C. (2012). Symmetric hyperbolic systems in algebras of generalized functions and distributional limits. *Journal of Mathematical Analysis and Applications*, *388*(2), 1166-1179. https://doi.org/10.1016/j.jmaa.2011.11.014

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

ABDOURAHMAN HAMAN ADJI had set the problem investigated and proposed completely the methodology of the research conducted within the whole work with all computations and verifications, presenting the main idea of the presentation of the paper.

MOMO ZANGUIM Suzie, and **SHANKISHVILI Lamara Dmitrievna** verified some computations realized within the whole work, established links with some papers related to distributional solutions for some singular linear differential equations in some particular distributional spaces.

Sources of funding for research presented in a scientific article or scientific article itself

The realization of this research work was partially started at the Department of Mathematical Analysis of the Kharkov State University in Ukraine. It has been followed at the Chair of Differential and Integral Equations of the Faculty of Mechanics and Mathematics at the State University of Rostov-on-Don in the Russian Federation and finalized at the University of Ngaoundere in Cameroon. The premium for the modernization of research from the Ministry of Higher Education of Cameroon financed this work.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.