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Numerical computation of buckling loads for variable stiffness bars under thermomechanical loading

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Abstract— This study investigates the buckling of an elastic bar with a linearly varying rectangular cross-section under thermomechanical loading. The research focuses on such straight bars' stress-strain state and stability, addressing large displacement scenarios using a nonlinear system of ordinary differential equations. The mathematical model considers axial and transverse displacements, rotation angles, and internal force factors, creating a variable temperature field along the length of the bar. A numerical solution is obtained using MathCad's built-in computational tools, enabling precise determination of displacement components, internal forces, and bending moments. The numerical calculation determines the value of the first critical force, which is essential for calculating the strength of the structure. The critical load is determined through numerical iteration from the compressive load.

Keywords— buckling, variable cross-section bar, thermomechanical loading, differential equations, numerical analysis, shooting method, critical force.

I. Introduction

The mathematical model of an elastic beam with a variable cross-section is studied in the works of [1]. Using the three-dimensional theory, the fields of displacements, deformations, and stresses are expanded into an orthogonal double Fourier-Legendre series concerning the variables of the cross-sectional thickness and width. All terms in the series, except the first, are neglected. The case where the variable cross-section degenerates into a linearly varying segment or a point is considered. The initial boundary value problem is analyzed, and the existence and uniqueness of solutions are demonstrated.

The deformation of a long thermally elastic beam with a rectangular cross-section under mixed boundary conditions is examined in [2], [3], [4]. Mixed boundary conditions involve a combination of prescribed temperature and displacement at the beam's ends. The beam's deformation is analyzed using the boundary integral method, which accounts for the combined effects of thermal and mechanical loads. The analysis also considers the thermally elastic properties of the material, including temperature-dependent thermal conductivity. The boundary integral formulation involves dividing the beam's surface into elements, representing the boundary conditions using integral equations, and solving the resulting system of equations to determine the beam's deformation. This study provides insights into the deformation analysis of long thermally elastic beams under mixed boundary conditions.

Articles [5], [6], [7] investigate the equilibrium of compressed beams with variable cross-sections. The study focuses on beams with varying thicknesses, where a fourth-order differential equation with variable coefficients describes deformation under longitudinal compression. The solution to this problem provides insights into the equilibrium behavior of the beam, accounting for variations in its transverse dimensions. It is shown that the variability of the beam's cross-sectional width significantly affects its equilibrium. The differential equation captures the complex interactions between the beam's geometry and its mechanical response during compression. This research contributes to a deeper understanding of the equilibrium of beams with non-uniform cross-sections. The findings have potential applications in various engineering fields, including structural mechanics and materials science.

Previous research has laid significant groundwork for buckling analysis under isolated mechanical or thermal loads. For instance, [8] employed finite element analysis (FEA) to study critical buckling loads in steel frames with varying slenderness ratios, demonstrating that numerical results align

with Euler load predictions for both sway and non-sway configurations. Similarly, [9] emphasized the importance of exact analytical solutions as benchmarks for validating numerical methods, compiling exact buckling solutions for columns, beams, and plates. These studies underscore the necessity of integrating numerical and analytical approaches, particularly for complex geometries. Recent advances in nonlinear solution techniques, such as the homotopy analysis method (HAM) applied by [10] to Euler columns with elastic restraints, further highlight the utility of advanced analytical tools in resolving boundary condition dependencies.

The challenge intensifies when geometric non-uniformity intersects with thermomechanical coupling. Variable cross-sections induce spatially dependent stiffness, complicating governing differential equations, while thermal gradients alter material properties (e.g., Young's modulus, thermal expansion coefficient) and generate non-uniform axial stresses. Although studies like [11] (post-buckled nonlinear beams on elastic foundations) and [12] integral formulation for beams with arbitrary boundary conditions have addressed nonlinearities in prismatic systems, their extension to non-prismatic rods under thermal loading remains sparse. [13] advanced this field by discretizing non-prismatic columns into prismatic segments on nonuniform elastic foundations, yet their work did not account for temperature effects. Furthermore, existing literature often decouples thermal and mechanical contributions, neglecting synergistic interactions that may precipitate premature instability. This oversight is critical in applications like reusable spacecraft or nuclear reactors, where cyclic thermal and mechanical loads demand robust predictive models.

This article addresses these gaps by investigating the buckling of variable cross-section rods under combined thermomechanical loading. Three primary objectives guide the study: 1. developing a generalized mathematical model that integrates temperature-dependent material properties, geometric tapering, and elastic foundation interactions; 2. employing a hybrid analytical-numerical approach—combining segmented exact solutions [13] with nonlinear FEA [8]—to resolve critical buckling loads and mode shapes, and 3. validating results against exact benchmarks [9] and post-buckled nonlinear solutions [11] to ensure accuracy. The proposed methodology not only accounts for arbitrary boundary conditions and foundation stiffness but also quantifies the role of thermal gradients in modulating buckling thresholds.

This work advances structural stability theory by reconciling geometric complexity with thermomechanical coupling and provides practical insights for optimizing lightweight, high-performance systems. The findings are particularly relevant to industries requiring resilience under extreme operational conditions, such as aerospace, energy, and automotive engineering, where variable cross-section rods are increasingly prevalent.

The article [14], [15], [16], [17] reports on research findings that encompass theoretical calculations, computer simulations, and experimental studies concerning displacements, internal force factors, and temperature fields during the plane deformation of rods. Specifically, it investigates the temperature problem associated with a rectangular cross-section bar. The one-dimensional unsteady heat conduction differential equation is solved within the MathCad system, considering first and third-kind boundary conditions, including those established experimentally for non-stationary scenarios. The numerical calculations computed in MathCad align well with experimental outcomes, with a maximum error of 17% noted at the end of the heating process at t =440 s. Additionally, the coordinates of the reduced center of gravity under uneven temperature distribution are determined, revealing that an increase in the temperature gradient affects these coordinates regardless of the temperature change law. The simulations conducted in MathCad have reduced the workload by about 20% when analyzing factors influencing internal stresses and temperature fields during the rods' plane deformation.

Meanwhile, papers [18], [19] explore variable cross-section bars. These bars lack uniformity in length and are subjected to a variable longitudinal force distributed along their axis. The discussion includes the loss of stability of the bar's straight equilibrium shape, as a curved shape may also occur. The critical relationship between stiffness and longitudinal force is derived from applying an integral representation of the original stability equation with variable coefficients, using a similar equation with constant coefficients as a reference. This integral representation contains the Green's function of the original equation, defined through perturbation methods. The numerical results obtained from these formulas have been compared with exact solutions for the stability equations in various specific scenarios.

Paper [20] presents general computer-aided (CA) way for optimization of bending beams that is an example of using the CA capabilities to obtain a large quantity of data to receive new qualitative information. The adopted model for the bending beam is an n-lumped beam unrelated to how big is n.

II. The Fundamental Equations of Straight Column Deformation

Geometrically nonlinear problems of planar deformation of the bars are discussed in the following publications, [14], [21], [22]. A fundamental system of equations has been obtained for the bars ($y/\rho_0 < 1$, ρ_0 is the straight column's axis's initial curvature; y - coordinate in the cross-section thickness) under thermomechanical loading. By passing to the limit $\rho_0 \to \infty$, he differential equations for the straight bars are obtained, [21].

We assume that the temperature field is one-dimensional and varies along the straight bar's axis. Let the curvature of the thermally elastic line after deformation be denoted by ρ. The angle of inclination of the sides of the thermally elastic line following deformation with respect to the *z*-axis will be denoted by θ. The displacement along the *z*-axis is denoted by

w, and the displacement along the *y*-axis is denoted by *v*. It is clear that.

$w=w(z), v=v(z), \rho=\rho(z), \theta=\theta(z),$

The system of fundamental equations for the deformation of a thermally elastic straight bar in the plane has the following form [21]:

Geometric equations:

$$\frac{dv}{dz} = (1+\varepsilon_0)\sin\theta$$
$$\frac{dw}{dz} = (1+\varepsilon_0)\cos\theta - 1 \quad (1)$$
$$\frac{d\theta}{dz} = \chi_x$$

Equations of statics:

$$\frac{dM}{dz} = (1+\varepsilon_0)(H\sin\theta - R\cos\theta - m)$$
$$\frac{dR}{dz} = -(1+\varepsilon_0)q_y \quad (2)$$
$$\frac{dH}{dz} = -(1+\varepsilon_0)q_z$$

where $\varepsilon_0$is the deformation of the thermally elastic line, χ_x is the characteristic of the change in curvature, the bending moment is denoted by *M*, the internal force components concerning the *y* and *z* axes are denoted by *R* and *H*, respectively q_y, and q_z are the distributed external forces relative to the y and z axes, respectively, and m is the intensity of the bending moment. Distributed external forces relative to the y and z axes, respectively, and m is the intensity of the bending moment.

If the material's elasticity modulus *E* is constant, then the deformation ε_0 of the thermally elastic line and the characteristic parameter of the change of the curvature χ_x is determined from the following relationships:

$$\varepsilon_0 = \frac{N}{A^*} + \frac{1}{A^*}\cdot E\int \varepsilon^T dA \quad (3)$$
$$\chi_x = \frac{M}{I_x^*} + \frac{E}{I_x^*}\int \varepsilon^T \cdot y \cdot dA, \quad (4)$$

where A^* is the bar's generalized cross-sectional area, I_x^* is the cross-section's generalized moment of inertia, *N* is the cross-sectional normal force, and ε^T is thermal deformation.

The normal force in the cross-section is determined as [14], [15]:

$$N=H\cos\theta+R\sin\theta \quad (5)$$

The generalized cross-sectional area A^* and the generalized moment of inertia I_x^* are calculated using the following formulas, [21]:

$$A^* = \int E(T)dA,\ I_x^* = \int y^2 E(T)dA, \quad (6)$$

The thermal deformation is given by $\varepsilon^T = \alpha(T-T_0)$, where the linear expansion coefficient is denoted by α, initial temperature is denoted by T_0 , and T represents the temperature at the specified time.

For external loads q_y=0 and *m*=0, we obtain the equations for the buckling of bars. For a rectangular linearly variable cross-section of a bar, see Figure 1, the system of fundamental equations (1), (2) takes the form:

$$\frac{dv}{dz} = (1+\varepsilon_0)\sin\theta$$
$$\frac{dw}{dz} = (1+\varepsilon_0)\cos\theta - 1$$
$$\frac{d\theta}{dz} = \chi_x \quad (7)$$
$$\frac{dM}{dz} = (1+\varepsilon_0)(H\sin\theta - R\cos\theta)$$
$$\frac{dR}{dz} = 0$$
$$\frac{dH}{dz} = -(1+\varepsilon_0)q_z$$

The deformation of the thermally elastic line and the radius of curvature after deformation are determined by the following formulas:

$$\varepsilon_0 = \frac{N}{EA(z)} + \varepsilon^T \quad (8)$$
$$\rho = \frac{1+\varepsilon_0}{\chi_x}, \quad (9)$$

where the bar's cross-sectional area is calculated according to Figure 1:

$$A(z) = ab\left[\left(\frac{b_0}{b}-1\right)\frac{z}{l}+1\right] \quad (10)$$

where a-width of rectangular cross-section.

According to formula (4), the characteristic parameter of the curvature change is determined as follows:

$$\chi_x = \frac{12M}{Eab^3\left[\left(\frac{b_0}{b}-1\right)\frac{z}{l}+1\right]^3} + \frac{E}{I_x^*}\int \varepsilon^T \cdot y \cdot dA \quad (11)$$

The temperature deformation ε^Tdoes not change concerning the y coordinate. Accordingly, in relationship (11), the second term on the right-hand side is zero relative to the axis passing through the centroid of the section.

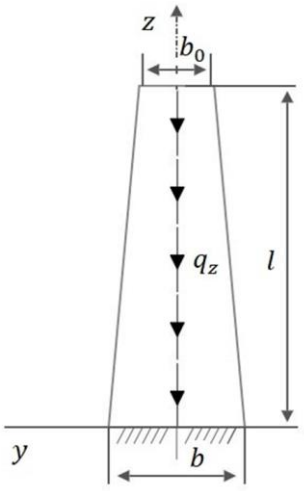


Fig. 1 A straight column with a variable cross-section.

The nonlinear differential equations (7), along with formulas (5), (8), and (11), form a closed system of equations. To integrate the differential equation system, boundary conditions depend on the type of fixation at the ends of the bar. For the rigid support $v=0$, $w=0$, $\theta=0$, and for the free end $H=0$, $R=0$, $M=0$.

III. Numerical Calculation

The nonlinear boundary problem can be effectively solved using numerical methods. For numerical calculations, we introduce dimensionless quantities:

$$\bar{z}=\frac{z}{l};\ \bar{q}_z=\frac{q_z}{q_*};\bar{\chi}_x,=\chi_x l;\ \bar{v}=\frac{v}{l};\ \bar{w}=\frac{w}{l};\ \bar{\rho}=\frac{\rho}{l};$$
$$\bar{N}=\frac{N}{q_* l};\ \bar{y}=\frac{y}{b};\ \bar{M}=\frac{M}{q_* l^2};\ \bar{H}=\frac{H}{q_* l};\ \bar{R}=\frac{R}{q_* l};\bar{T}=\frac{T}{T_0}$$

$$\bar{A}(z)=\frac{A(z)}{A_0}\ ;\ \lambda_1=\frac{q_* l}{Eab};\ \lambda_2=\alpha T_0;\ \lambda_3=\frac{12 q_* l^3}{Eab^3},$$

Where q_*- is the highest value of the distributed load, and l- is the bar's length before deformation.

The equation system (7), (5), (8), and (11) can be written in dimensionless quantities as follows:

$$\frac{d\bar{v}}{d\bar{z}}=(1+\varepsilon_0)\sin\theta,$$
$$\frac{d\bar{w}}{d\bar{z}}=(1+\varepsilon_0)\cos\theta-1,$$
$$\frac{d\theta}{d\bar{z}}=\overline{\chi_x},$$
$$\frac{d\bar{R}}{d\bar{z}}=0,$$
$$\frac{d\bar{H}}{d\bar{z}}=-(1+\varepsilon_0)\bar{q}_z,$$
$$\frac{d\bar{M}}{d\bar{z}}=(1+\varepsilon_0)(\bar{H}\sin\theta-\bar{R}\cos\theta)$$
$$\bar{N}=\bar{H}\cos\theta+\bar{R}\sin\theta \tag{13}$$
$$\varepsilon_0=\frac{\lambda_1\bar{N}}{\left(\frac{b_0}{b}-1\right)\bar{z}+1}+\lambda_2(\bar{T}-1) \tag{14}$$
$$\bar{\chi}_x=\frac{\lambda_3\bar{M}}{\left[\left(\frac{b_0}{b}-1\right)\bar{z}+1\right]^3} \tag{15}$$

The system of equations (12) can be represented in the following manner in vector-matrix form:

$$Y'=F(y,\bar{q}),\quad \text{where } y=(\bar{v},\bar{w},\theta,\bar{R},\bar{H},\bar{M},)^T \tag{16}$$

IV. Calculation and Results Analysis

For performing numerical calculations in the MathCad system, the following designations have been introduced:

$y_1=\bar{v}; y_2=\bar{w}; y_3=\theta; y_4=\bar{R}; y_5=\bar{H}; y_6=\bar{M};$

Then, a system of equations (12),(14),(15))will have the following form:

$$y_1'=(1+\varepsilon_0)\sin y_3$$
$$y_2'=(1+\varepsilon_0)\cos y_3-1,$$
$$y_3'=\frac{\lambda_3 y_6}{\left[\left(\frac{b_0}{b}-1\right)\bar{z}+1\right]^3} \tag{17}$$
$$y_4'=0$$
$$y_5'=-(1+\varepsilon_0)\bar{q}_z$$
$$y_6'=(1+\varepsilon_0)(y_5\sin y_3-y_4\cos y_3)$$
$$\varepsilon_0=\lambda_1\frac{y_5\cos y_3+y_4\sin y_3}{\left(\frac{b_0}{b}-1\right)\bar{z}+1}+\lambda_2(\bar{T}-1) \tag{18}$$
$$\bar{\chi}_x=\frac{\lambda_3 y_6}{\left[\left(\frac{b_0}{b}-1\right)\bar{z}+1\right]^3} \tag{19}$$

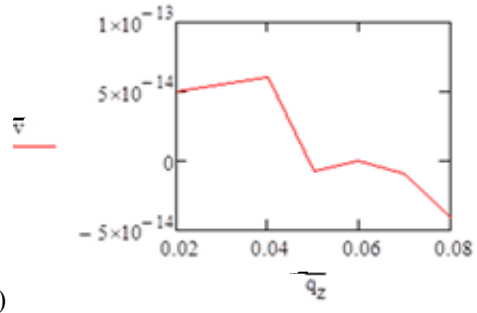
In this case, the boundary conditions have the form: $\bar{z}=0$; $y_1=0$, $y_2=0$ $y_3=0$; $\bar{z}=1$; $y_4=0$, $y_5=0$ $y_6=0$. As a test

item, the transverse bending of a cantilever bar with a consistent cross-section, with a load distributed with a constant intensity, was calculated. The results of the numerical calculation are exactly in line with the known analytical solution.

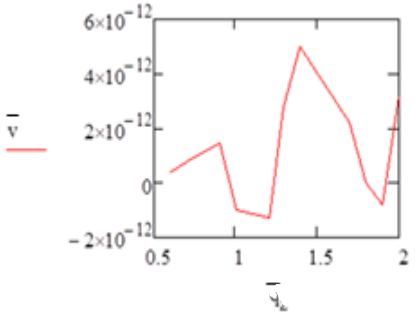
The variable rectangular cross-section bar was analyzed for stability under constant-intensity thermomechanical loading for the case of b_0/b=0,1 at various temperature values.

When solving a nonlinear boundary value problem in MathCad, the missing boundary conditions at the beginning of the integration interval (shooting parameters) are determined by the built-in function *sbval*, [23].

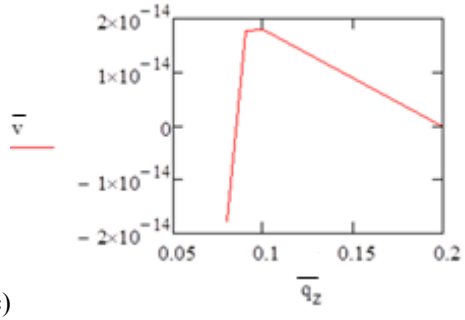
The convergence of the iterative process significantly depends on the accuracy of selecting the initial approximation for the shooting parameters. The solution of the linear problem was used as the zero approximation of the shooting parameters. The convergence of the iterative process at each load step took no more than 20 seconds. The problem is solved step by step for small load increments. With arbitrary selection of shooting parameter values, the function *sbval* failed to find a solution. To calculate $\bar{y}_i(\bar{z}), i = 1,...,6$ over the entire interval, an additional Cauchy problem is solved using the Runge-Kutta method with a fixed pitch *rkfixed*, [23]. One of the outcomes of the presented work is that it shows the possibility of solving a nonlinear boundary value problem of bar deformation and determining the critical load using standard programs (built-in functions) in MathCad.



a)



b)



c)

Fig. 2 Transverse displacement dependence on load at different heating temperatures: a) 40°C, b) 60°C, c) 100°C.

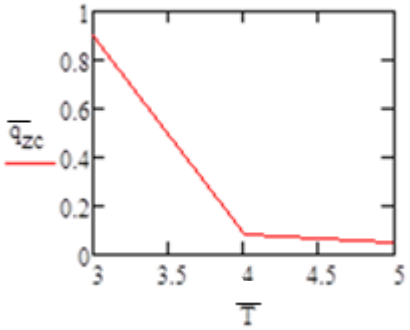


Fig. 3 Dependence of critical load on dimensionless temperature.

Figure 2, a, b, c illustrates the dependence of transverse displacement ($\bar{v}$) of the free end of the bar on longitudinal load ($\bar{q}_z$) at different levels of heat. The graphs in Figure 2 allow us to determine the critical load values.

For example, Figure 2 c illustrates the relationship between the transverse displacement ($\bar{v}$) of the free end of the bar and the load parameter ($\bar{q}_z$) at a given temperature of 100 °C. Numerical calculations show that $\bar{q}_z = 0.075$ is the limit point at which $\frac{\partial \bar{v}}{\partial \bar{q}_z} \to \infty$. As is known [24], different solutions do not intersect at limit points, but when passing through this point, stable equilibrium becomes unstable. In this case, the limit point of the initial equilibrium form is critical. The critical load corresponding to the critical point of deformation was established, at which $\frac{\partial \bar{v}}{\partial \bar{q}_z} \to \infty$. Also, at critical points the monotonic increase in transverse displacement transitions to a decrease. This is how the first critical points and corresponding critical loads were determined for each set temperature.

Figure 3 illustrates the dependence of the first critical load on temperature, from which it is evident that the value of the first critical load decreases as temperature increases.

V. Conclusions:

1. Nonlinear differential equations are presented for analyzing the stability of a straight bar under thermomechanical loading. Boundary conditions are formulated.

2. The nonlinear boundary value problem for the stability of a bar with a linearly varying rectangular cross-section is solved numerically using the shooting method. Numerical calculations are performed, and the critical load value is determined.

3. Numerical calculations establish the dependence of the first critical load on temperature. It is found that the value of the first critical load decreases with increasing temperature.

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Declaration of Generative AI and AI-assisted Technologies in the Writing Process

The authors wrote, reviewed and edited the content as needed and verifies that none utilised artificial intelligence (AI) tools were used. The authors take full responsibility for the content of the publication.

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Irakli Tavadze: Validation, Data Curation, Writing – Review & Editing.

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Conflict of Interest

The authors declare no conflict of interest.

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