

A Suggested Modification of Millionshchikov's Zero-Fourth Cumulant Hypothesis

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Abstract—Over a few decades researchers, working in the field of turbulence, are examining the zero-fourth cumulant model, proposed by Millionshchikov. Numbers of modifications have been suggested. In some cases the problem of negative energy at the initial stage is observed and further modification is suggested. Some have tried to point out that the value of fourth cumulant may not remain zero all through the hierarchy of eddies vis-à-vis over all length scales and suggested different modification.

In the present paper we have tried to suggest a new model. Its viability has also been examined to some extent. In the final period decay of isotropic turbulence, the smallest eddies of the largest wave number only are active. Viscous dissipation of turbulent energy in the form of thermal energy takes place in this final period. The largest

wave number is defined as $k_\infty = \left(\frac{\nu^3}{2}\right)^{\frac{1}{4}}$. In the final period decay,

neither inertia force nor pressure gradient takes part in the decay process. But these two terms, in fact, are the sources of third order moment.

Keeping this in mind we have tried to develop a model for fourth cumulant, expressed as $\tilde{K}_4 = \tilde{u}^4 - (\tilde{u}^2)^2$ in wave number space.

In the present case we would use the symbol $\tilde{u}$ only, since in isotropy $\tilde{u}_1^4 = \tilde{u}_2^4 = \tilde{u}_3^4$ and $\tilde{u}_1^2 = \tilde{u}_2^2 = \tilde{u}_3^2$. With targets given above, we present a model of $\tilde{K}_4$.

Keywords—Millionschikov's hypothesis, zero-fourth cumulant hypothesis, homogeneous and isotropic turbulence, Hermite polynomial.

I. INTRODUCTION

IN “zero fourth cumulant hypothesis” of Millionshchikov [1] second and fourth order moments are related as in normal distribution. The hypothesis goes also by the name of quasi-normal hypothesis. Millionshchikov has proposed that the two point distribution of simultaneous velocity amplitudes in a turbulent flow is quasi-normal. A lot of research has been done on this hypothesis. Research papers of legendary mathematicians like Heisenberg [2], Obukhov [3], Batchelor [4], Chandrasekhar [5] and Reid and Proudman [6] mentioning. Experimental results have also been analyzed by Batchelor and Uberoi [7] to verify the authenticity of this hypothesis.

Heisenberg has extended the hypothesis to the distribution of Fourier coefficients in two times. On the other hand, Reid and Proudman examined the hypothesis in the decay of homogenous turbulence using three point simultaneous velocity fluctuations. Chandrasekhar has studied the hypothesis in hydromagnetic turbulence. Along with the experimental works of Batchelor and Uberoi, the theoretical works conducted by Ghosh [8] may be mentioned.

Recently, there is a spurt of work considering the viability or otherwise of this hypothesis. Bos & Rubenstein [9], Chen, H., Herring, J.R., Kerr, R.M., & Kraichnan R.H [10], Chen, S. & Kraichnan, R.H. [11], Mazumdar, H.P. & C. Mamaloukas [12] and Mamaloukas [13] have studied the hypothesis in detail.

Suggestions have also been forwarded from time to time to modify the hypothesis. Ogura [14] has suggested some modification. Mirabel [15] has incorporated some modification to remove the discrepancies presented in the suggestion made by Ogura. The suggestion vis-à-vis objection regarding the viability of Millionshchikov hypothesis forwarded by Kraichnan [16] is note-worthy. In the present paper we shall try to suggest a modification of the “zero-fourth cumulant hypothesis” of Millionshchikov. We would confine our study to one-point one-time correlations in isotropic turbulence, and in wave number space as well.

II. EQUATIONS OF MOTION IN ISOTROPIC TURBULENCE

We have studied homogenous and isotropic turbulence. It is observed that in the final period of decay only the smallest eddies are active and viscous dissipation is the only mechanism through which turbulent energy decays in the form of thermal energy. The size or the length scale of these

smallest eddies is given as $\lambda_0 = \left(\frac{\nu^3}{2}\right)^{\frac{1}{4}}$. In wave number space

these smallest eddies have the largest wave number, as

$k_\infty = \left(\frac{\nu^3}{2}\right)^{\frac{1}{4}}$. In the final period of decay neither inertial force

nor pressure gradient takes part in the decay process. But these

two terms, in fact, are the sources of third order moment. We now try to examine this point from the Navier-Stokes equation of homogenous and isotropic turbulence. The unit density, incompressible homogenous and isotropic fully developed turbulent flow in absence of external force is governed by the following equations.

$$\frac{\partial u(x,t)}{\partial t} + [u(x,t) \cdot \nabla u(x,t) - \nabla p(x,t)] = \nu \Delta u(x,t) \quad (1)$$

$$\nabla u(x,t) = 0$$

Kraichnan & Panda [17] observe that in turbulent flow the variance of the nonlinear term in the Navier-Stokes equations i.e. $[u(x,t) \cdot \nabla u(x,t) - \nabla p(x,t)]$ is on average smaller than what would be expected from a Gaussian estimate. More precisely, if one constructs a flow field consisting of random statistically independent Fourier modes exhibiting the same energy spectrum as the turbulent flow considered, the variance of the nonlinear term will be larger than that of the original field. This depletion of nonlinearity is the result of a self-organization process of the turbulent flow, a process which is, itself, due to the nonlinear term in the Navier-Stokes equations. Kraichnan and Panda give importance on the velocity-vorticity alignment in turbulent flows. They show it to be one expression of a more general, underlying property of nonlinear systems. We consider that this depletion of nonlinearity is an important feature of turbulent flows, since the nonlinearity of the N-S equations is the heart of the turbulence problem.

The nonlinear term of the Navier-Stokes equations is a vector and its mean value is zero in isotropic turbulence. It is important to study the strength of the fluctuations of the nonlinear term and its depletion.

In order to examine the strength of the fluctuations of the nonlinear term and its depletion, we focus on the nonlinearity spectrum, which we shall define below. This spectrum measures the strength of the fluctuations of the nonlinear term as a function of scale, just like the energy spectrum does for the strength of velocity fluctuations. Whereas the characterization of the energy spectrum has received an enormous amount of attention in the field of turbulence research, only very few investigations consider the nonlinearity spectrum. To our knowledge, only the works by Chen, Herring & Kraichnan, Nelkin & Tabor [18] and Ishihara, T., Kaneda, Y., Yokokawa, M., Itakura, K., & Uno, A. [19] have considered this quantity. Chen et al. performed low resolution Direct Numerical Simulations and compared their results to the Direct Interaction Approximation (DIA). No information was obtained on the inertial range behavior of this quantity, since the Reynolds number was too low in their simulations. Higher Reynolds numbers could in principle be obtained by using the DIA, but physically incorrect behavior is observed by Kraichnan in the inertial range dynamics of the original Eulerian DIA. Nelkin & Tabor considered only the scaling of the potential part of the advection term, assuming that the full nonlinear term scales as its potential part. Only the high

resolution simulations by Ishihara et al. give an idea on the inertial range scaling of several fourth order spectra.

It is shown by Bos & Rubenstein that, in the inertial and dissipation range, the nonlinearity spectrum is given by

$$w(k) = \tilde{u}^{\frac{2}{3}} \tilde{u}^{\frac{3}{2}} k^{\frac{1}{3}} f(k\eta).$$

For very high Reynolds numbers $\tilde{u}$ is the root-mean-square (rms) velocity fluctuation. The function $f(k\eta)$ tends to a constant value in the inertial range and its value is approximately 0.8 times the value of its Gaussian estimate.

The total depletion of nonlinearity is found to exhibit some sub-Gaussian behavior. Gryanik & Hartmann [20] have observed from CBL data that the Millionshchikov hypothesis of quasi-normal (Gaussian) distribution of the one-point fourth-order moments fails for convective boundary layer conditions. This is because the effect of the semi organized coherent structures (plumes) leads to skewed distributions; the third-order moments are non-zero. It must be connected with a certain order in the flows, but how this manifests itself in an instantaneous flow field cannot be guessed from the statistical considerations presented here. The nonlinear term consists of two parts: the advection term and the pressure gradient term. The pressure spectrum, (Gotoh & Fukayama [21]) $\pi(k)$

scales approximately as $\pi(k) \sim 2^{\frac{4}{3}} k^{\frac{1}{3}} f(k\eta)$ and the pressure gradient spectrum scales as $\nabla \pi(k) \sim 2^{\frac{3}{4}} k^{\frac{7}{3}} f(k\eta)$

It may be noted, however, that this scaling appears only at relatively high Reynolds number, compared to the appearance of K41 [22] scaling for the energy spectrum. It is shown by Bos & Rubenstein that at large Reynolds numbers the mean square nonlinearity is proportional to the Gaussian value, the ratio being 0.65. The variance of the nonlinearity is therefore dominantly determined by the advection term. The depletion of nonlinearity implies hereby directly a depletion of the sweeping compared to the kinematic sweeping induced by a field consisting of independent Fourier modes. Recent works of Hans C. Eggers and Martin Greinerb [23] and Servidio, S., Matthaeus, W. H., & Dmitruk, P. [24] are worth mentioning.

In this context we refer to the work by Chen, H., Herring, J.R., Kerr, R.M., & Kraichnan, R.H., which discusses the possibility of a reduction of sweeping in turbulence. They argue that third order cumulants are small but nonzero quantities. But in the pre-viscous dissipation ranges, dominated by large eddies of smaller wave number k triple order interaction are so great in number that their total contribution cannot be ignored to get to the realistic analysis of turbulence. Chen, H., Herring, J.R., Kerr, R.M., & Kraichnan, R.H. argue that a complete reduction of sweeping is improbable for stochastically forced turbulence. Their arguments are not in disagreement with the experimental results, obtained thereafter.

The dependence of the large and small scales and the sweeping is influenced, as estimated, by purely kinematic

arguments and is partially but definitely not completely suppressed. In this light, the depletion of nonlinearity can also be interpreted as a reduction of Eulerian acceleration, suggesting a larger Eulerian coherence for turbulence than for advection by random Fourier modes. The possible link of this enhanced coherence with inertial range and dissipation range intermittency is not clear at present. The super-Gaussian values of the large-scales of the nonlinearity spectrum were shown to be related to the non-Gaussianity of the Reynolds-stress-fluctuation spectrum. The physical importance of this relation for the dynamics of turbulent flow seems to deserve further research.

We mention here that a similar picture (large-scale super-Gaussian behavior and sub-Gaussian inertial range and dissipation range behavior), was observed in the depletion of advection (Bos et al.), where the inertial range scaling of the advection spectrum also displayed a constant reduction with respect to its Gaussian value.

III. THE PRESENT MODEL

Given above the background of study of nonlinearity vis-à-vis third order moment of turbulent flow we would now try to present a new model of one point one time moment in homogenous and isotropic turbulence of incompressible fluid with unit density. The main considerations used for constructing the model may be enumerated as below.

1. At the viscous dissipation range i.e where $k \rightarrow \infty$, turbulence energy dissipates only in the form of thermal energy. In this range interaction among eddies of different size ceases to exist. Distribution of Fourier components of velocity fluctuation is assumed Gaussian. Zero-fourth cumulant hypothesis is strictly followed in this range.

2. As wave number becomes smaller and smaller a part of turbulence energy, not the whole, dissipates as thermal energy and the residual part is involved in interacting with Fourier modes of velocity fluctuations and cascading from smaller wave number to larger wave number. As k progresses to the smallest number, it crosses universal equilibrium range and large eddy range one after the other. In this passage the ratio between dissipated thermal energy and internal energy involved in interaction and cascading becomes less and less. In these ranges third order interaction is non-zero which indicates the strict presence of third order cumulant, however small it may be.

In consideration of the above points, we now present the following model, It may be noted that we are restricting our study to wave number space only of isotropic turbulence.

1. $\tilde{K}_4 = \overline{\tilde{u}^4} - \left(\overline{\tilde{u}^2}\right)^2 = 0$ for $k = k_\infty$, and
2. $\tilde{K}_4 = \overline{\tilde{u}^4} - \left(\overline{\tilde{u}^2}\right)^2 > 0$ for $k \ll k_\infty$

With targets given above we present a near-Gaussian model of distribution of u . The probability density of the proposed model is given as below.

$$\omega = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}} \left\{ 1 + U \left(1 - \frac{k}{k_\infty} \right) \sum_{i=3}^{\infty} \frac{c_i}{i!} H_i \left(\frac{u}{\sigma} \right) \right\} \quad (2)$$

Here $U = 0$ for $k \approx k_\infty$, and $U=1$ for $k < k_\infty$. In addition c_i 's are given as below,

$$c_3 = \frac{\overline{x^3}}{\sigma^3}, c_4 = \frac{\overline{x^4}}{\sigma^4} - 3, c_5 = \frac{\overline{x^5}}{\sigma^5} - 10 \frac{\overline{x^3}}{\sigma^3} \quad (3)$$

H is the Hermite polynomial. The first three Hermite polynomials in the above expression are given as below,

$$\begin{aligned} H_3(x) &= 8x^3 - 12x \\ H_4(x) &= 16x^4 - 48x^2 + 12 \\ H_5(x) &= 32x^5 - 160x^3 + 120x \end{aligned} \quad (4)$$

ω being a probability density, it becomes equal to unity, if summed up over all Fourier modes. So, integrating over the whole range of u we get the following condition.

$$\int \omega du = 1 \quad (5)$$

ω becomes a Gaussian probability density ω' in viscous dissipation range and fourth cumulant becomes equal to zero satisfying zero-fourth cumulant hypothesis of Millionshchikov. But in the other ranges it is not so.

For viscous dissipation range, when $k \rightarrow \infty$ we have the following expression for ω ,

$$\omega' = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}} \quad (6)$$

ω' in (6) is the probability density of Gaussian distribution, that supports zero-fourth cumulant hypothesis. For other ranges ($k \ll k_\infty$) we shall use the expression ω'' for ω' to show the difference of these two probability densities. ω'' is given as below,

$$\omega'' = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}} \left\{ 1 + \sum_{i=3}^{\infty} \frac{c_i}{i!} H_i \left(\frac{u}{\sigma} \right) \right\} \quad (7)$$

It may be noted that the near-Gaussian probability, expressed by ω'' is far from Gaussian ω'

Integrating (7) over the whole range of values of u we get as below,

$$\omega'' du = \int \left\{ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}} \left\{ 1 + \sum_{i=3}^{\infty} \frac{c_i}{i!} H_i \left(\frac{u}{\sigma} \right) \right\} \right\} du \quad (8)$$

From (8) we can find the third order cumulant as below,

$$\int u^3 \omega'' du = \int u^3 \left\{ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}} \left\{ 1 + \sum_{i=3}^{\infty} \frac{c_i}{i!} H_i \left(\frac{u}{\sigma} \right) \right\} \right\} du \quad (9)$$

The third order cumulant is zero in Gaussian distribution. But for the present near-Gaussian distribution (9) we have the following expression for third order cumulant, after using (6)

$$\int u^3 \omega'' du = \int u^3 \left\{ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}} \sum_{i=3}^{\infty} \frac{c_i}{i!} H_i \left(\frac{u}{\sigma} \right) \right\} du \quad (10)$$

IV. CONCLUSION

It has been opined by Kraichnan that even if zero fourth cumulant theory of Millionschikov does not hold good in fully developed turbulence, third order interaction is weak. Only because of large number of these interactions their total impact cannot be ignored. As a result, there is a nonzero third cumulant indicating non-Gaussianity in distribution of velocity components. From this understanding it may be conjectured that velocity components are distributed in a near Gaussian pattern. In this paper we have considered one time one point moments in isotropic turbulence. In the background of the sets of weak third order interaction we have used Hermite polynomial to build a non-Gaussian probability density for ranges beyond viscous dissipation range.

We have ignored any change in the distribution of third order interaction in different ranges of turbulence. It may be observed that different non Gaussian probability densities may be considered by suitably truncating Hermite polynomials in their expressions. But the authenticity of such exercise would depend on experimental verification. This may be an open problem.

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