On minimum covering energy of semigraph

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Abstract—A compromise between the concept of graph and hypergraph is semigraph. This paper introduced the concept of minimum covering matrix and minimum covering energy of a semigraph *G*. The minimum covering energy is the summation of singular values of the minimum covering matrix. Upper and lower bounds for minimum covering energy are established and also derive some relationship between minimum covering energy and energy of semigraph *G*.

Keywords—Minimum covering matrix, minimum covering eigenvalue, minimum covering energy.

I. INTRODUCTION

Graph energy is the concept that stems from chemistry to approximate the total π-electron energy of a molecule. In chemistry, the conjugate hydrocarbon can be represented by a graph called a molecular graph. The study of graph energy for all arbitrary graphs was initiated by [1], in the year 1978 and defined as the sum of the absolute values of its eigenvalues. Motivated by I. Gutman's work on the energy of graphs many authors conceived of different types of graph energy like color energy, [2], [3], distance energy, [4], etc. on graph theory. In the year 2012, [5], introduced a new matrix, called the minimum covering matrix of a graph and its energy, and defined as follows:

Suppose $G(V,X)$ be a graph of order *n* and size *m*, with vertex set *V* and edges set *X*. Let *C* subset of *V* be the minimum covering a set of a graph *G*. The minimum covering matrix of *G* is the square matrix $A_c(G) = (a_{ij})$ of order *n*, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 1 & \text{if } i = j \text{ and } v_i \in C \\ 0 & \text{otherwise.} \end{cases}$$

And the minimum covering energy of *G* is then defined as

$$E_c(G) = \sum_{i=1}^{n} |\lambda_i|$$

where $\lambda_1, \lambda_2, \ldots.., \lambda_n$ are eigenvalues of the minimum covering matrix $A_c(G)$ of *G*.

Semigraphs are a kind of generalization of graphs. A semigraph G is defined by [6], as an ordered pair (V, X) where $V = \{v_1, v_2, \ldots, v_n\}$ is a non-empty (usually finite) set of elements called vertices of G, whereas $X = \{e_1, e_2, \ldots., e_m\}$ is a set of ordered *k*-tuples of distinct vertices, called the edges of G. Each edge consists of a *k*-tuple of vertices, for various values of $k \geq 2$, satisfying the following conditions:

a. Two edges have at most one vertex in common.
b. Two edges $(x_1, x_2, \ldots, x_p)$ and $(y_1, y_2, \ldots, y_q)$ are considered to be the same if *p* = *q* and either $x_i = y_j$ for $1 \leq i \leq p$ or $x_i = y_{p+1-i}$ for $1 \leq i \leq p$.

Recently, [7], introduced the *energy of semigraph* in the year 2019 and established the upper and lower bounds for the energy $E(G)$ of a Semigraph $G(V,X)$ of order n, size m, as

$$\sqrt{2\sum_{e\in X}\left(1^2+2^2+\ldots+k_e^{\,2}\right)}\le E(G)\le\sqrt{2n\sum_{e\in X}\left(1^2+2^2+\right.}$$

Where $e\in X$ be an edge of cardinality k_e+1 .

Further, [8], in 2021, introduced the concept of *Distance matrix and energy of semigraph*.

A set of vertices that covers all the edges is a vertex cover for a semigraph *G*. And a set of vertex with minimum cardinality covering all the edges of G is called its minimum covering. The minimum cardinality of a vertex cover is called vertex covering number and it is denoted by $\alpha_0=\alpha_0(G)$.

Here a new type of matrix is introduced, called the minimum covering matrix of a Semigraph. Further studies its singular values and energy.

In, [12], the authors introduced adjacency matrix of signed semigraph and derived some properties.

II. Minimum covering matrix and its energy of a semigraph

In the year 2017, [9], defined an adjacency matrix associated with a semigraph as

The adjacency matrix of a semigraph:

Let $G(V,X)$ be a semigraph with vertex set $V=\{v_1,v_2,\ldots\ldots,v_n\}$ and edge set $X=\{e_1,e_2,e_3,\ldots\ldots,e_m\}$. The Adjacency matrix $G(V,X)$ is a $n\times n$ matrix $A=[a_{ij}]$ defined as follows:

i. For every edge e_i of *X* of cardinality, say *k*, let $e_i=(i_1,i_2,i_3,\ldots\ldots,i_k)$ such that $i_1,i_2,i_3,\ldots\ldots,i_k$ are distinct vertices in V , for all $i_r\in e_i$; $r=1,2,\ldots,k$
   (a) $a_{i_1i_r}=r-1$
   (b) $a_{i_ki_r}=k-r$

ii. All the remaining entries of A are zero.

Thus, we defined the minimum covering matrix of a semigraph as follows

A. The minimum covering matrix of a semigraph:

If $G(V,X)$ be a semigraph order *n* and, size *m*. Let *C* be the minimum covering set, then the minimum covering matrix of *G* is the square matrix $A_{mc}(G)=(a_{ij})$ of order n, where

i. For every edge e_i of *X* of cardinality, say *k*, let $e_i=(i_1,i_2,i_3,\ldots\ldots,i_k)$ such that $i_1,i_2,i_3,\ldots\ldots,i_k$ are distinct vertices in V , for all $i_r\in e_i$; $r=1,2,\ldots,k$
   (a) $a_{i_1i_r}=r-1,$
   (b) $a_{i_ki_r}=k-r$

ii. $a_{ij}=1$ if $i=j$ and $v_i\in C$.

iii. All the remaining entries of *A* are zero.

B. The minimum covering energy of semigraph:

In [10], the author defined the energy of a general matrix (of any size) as the summation of the singular values of that matrix.

Thus, If $\sigma_1,\ \sigma_2,\ \ldots\ldots\ldots,\sigma_n$ be the singular values of the minimum covering matrix $A_{mc}(G)$ of the semigraph *G*, then the minimum covering energy of a semigraph denoted by $E_{mc}(G)$, is defined as the summation of its singular values. i.e.

$$E_{mc}(G)=\sum_{i=1}^{n}\sigma_i$$

We observe that, $A_{mc}(G)A'_{mc}(G)$ is a positive semidefinite matrix. So its eigenvalues $\lambda_1,\ \lambda_2,\ \ldots\ldots\ldots\ldots,\lambda_n$ are non-negative and therefore the singular values of $A_{mc}(G)$ are non-negative real numbers. Thus $E_{mc}(G)\ge 0$, equality holds if and only if the number of edges in *G* is zero.

The minimum covering energy of a semigraph is well defined, as if G' be a semigraph obtained by relabeling of the vertices of *G*, then $A_{mc}(G')A'_{mc}(G')$ is obtained by interchanging the rows and the corresponding columns of $A_{mc}(G)A'_{mc}(G)$. Hence the eigenvalues of $A_{mc}(G)A'_{mc}(G)$ and $A_{mc}(G')A'_{mc}(G')$ are the same, and so the singular values of G and G' are also the same.

III. Properties of minimum covering energy of semigraph

Theorem 1 Let $A_{mc}(G)$ is the minimum covering matrix of a semigraph G, and C is its minimum covering set. If $\lambda_1,\ \lambda_2,\ \ldots,\lambda_n$ are eigenvalues of $A_{mc}(G)A'_{mc}(G)$. Then

$$\sum_{i=1}^{n}\lambda_i=2\sum_{e\in X}\left(1^2+2^2+\ldots+k_e^{\,2}\right)+|C|$$

where the cardinality of an edge $e\in X$ of the semigraph is k_e+1 and $k_e\ge 1$.

Proof: In the minimum covering matrix $A_{mc}(G)$, corresponding to every edge $e\in X$ of cardinality k_e+1 ,

there is a sequence $\{1, 2, ..., k_e\}$ in the rows corresponding to the end vertices of that edge. And there are $|C|$ nos. of 1's in the diagonal of $A_{mc}(G)$. Thus every edge contributes $2\sum_{e}\left(1^2 + 2^2 + ... + k_e{}^2\right)$ and the diagonal elements contribute $|C| \times 1^2$ to the trace of $A_{mc}(G)A'_{mc}(G)$.
Therefore

$$trac(A_{mc}A'_{mc}) = 2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e{}^2\right) + |C| \times 1^2$$

Hence $$\sum_{i=1}^{n}\lambda_i = 2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e{}^2\right) + |C|$$

Theorem 2. The minimum covering energy $E_{mc}(G)$ of a semigraph *G*, is a square root of an even or odd integer according as $|C|$ is even or odd.

Proof: If $\sigma_1, \sigma_2, ...,\sigma_n$ be the singular values of the minimum covering matrix $A_{mc}(G)$ of the semigraph *G*, then

$$(\sigma_1 + \sigma_2 + ... + \sigma_n)^2 = \sum_{i=1}^{n}\sigma_i^2 + 2\sum_{i<j}\sigma_i\sigma_j$$

Thus $$[E_{mc}(G)]^2 = \sum_{i=1}^{n}\lambda_i + 2\sum_{i<j}\sigma_i\sigma_j$$

$$= 2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e{}^2\right) + |C| + 2\sum_{i<j}\sigma$$

$$= 2\left[\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e{}^2\right) + \sum_{i<j}\sigma_i\sigma_j\right]$$

$$E_{mc}(G) = \sqrt{2\left[\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e{}^2\right) + \sum_{i<j}\sigma_i\sigma_j\right] + |C}$$

Thus the minimum covering energy $E_{mc}(G)$ of a semigraph *G*, is a square root of an even or odd integer according as $|C|$ is even or odd.

Theorem 3. The minimum covering energy $E_{mc}(G)$ of a semigraph *G*, then

$$[E_{mc}(G)]^2 = |C| (\mathrm{mod}\, 2)$$

Proof: By Theorem 2**,** the minimum covering energy $E_{mc}(G)$ of a semigraph *G*, is a square root of an even or odd integer according to $|C|$ is even or odd. i.e.

$$E_{mc}(G) = \sqrt{2t + |C|}$$ where t is a positive integer.

$$[E_{mc}(G)]^2 = 2t + |C|$$

Thus $$[E_{mc}(G)]^2 = |C| (\mathrm{mod}\, 2)$$

IV. SOME BOUNDS ON MINIMUM COVERING ENERGY OF SEMIGRAPH

Theorem 4. For a semigraph *G* on *n* vertices and *m* edges,

$$\sqrt{2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^2\right) + |C|} \le E_{mc}(G) \le \sqrt{n\left[2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^2\right) + |C|\right]}$$

Proof: Let σ_i, $i = 1, 2, ..., n$ be the singular values of A_{mc}, and λ_i, $i = 1, 2, ..., n$ be the eigenvalues of $A_{mc}(G)A'_{mc}(G)$. By Cauchy Schwarz's inequality on two vectors $(\sigma_1, \sigma_2, ..., \sigma_n)$ and $(1, 1, ..., 1)$, we have

$$(\sigma_1 + \sigma_2 + ... + \sigma_n)^2 \le n\sum_{i=1}^{n}\sigma_i^2 = n\sum_{i=1}^{n}\lambda_i$$

Thus,

$$[E_{mc}(G)]^2 \le n\left[2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e{}^2\right) + |C|\right]$$

Again we have

$$[E_{mc}(G)]^2 = \left(\sum_{i=1}^{n}\sigma_i\right)^2 \ge \sum_{i=1}^{n}\sigma_i^2 = \sum_{i=1}^{n}\lambda_i$$

Thus,

$$[E_{mc}(G)]^2 \ge 2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e{}^2\right) + |C|$$

Hence

$$\sqrt{2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^2\right) + |C|} \le E_{mc}(G) \le \sqrt{n\left[2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^2\right) + |C|\right]}$$

Theorem 5. If *G* is a semigraph having *n* vertices and *m* edges, then

$$[E_{mc}(G)]^2 \ge 2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e{}^2\right) + |C| + n(n-1)\Delta^{1/n}$$

,
Where $\Delta = \det(A_{mc}A'_{mc})$.

Proof: Let σ_i, $i = 1, 2, ..., n$ be the singular values of $A_{mc}(G)$, then we have

$$[E_{mc}(G)]^2 = \left(\sum_{i=1}^{n}\sigma_i\right)^2 = \sum_{i=1}^{n}\sigma_i^2 + 2\sum_{i<j}\sigma_i\sigma_j = \sum_{i=1}^{n}\lambda_i +$$

As σ_i, $i=1,2,.....,n$ are non-negative, so $n(n-1)$ nos. of $\sigma_i\sigma_j$ are also non-negative number.

Therefore, applying $AM \geq GM$ on $n(n-1)$ nos. of non-negative numbers $\sigma_i\sigma_j$. We have

$$\frac{1}{n(n-1)}\sum_{i\neq j}\sigma_i\sigma_j \geq \left(\prod_{i\neq j}\sigma_i\sigma_j\right)^{\frac{1}{n(n-1)}} = \left(\prod_{i=1}^{n}\sigma_i^{2(n-1)}\right)^{\frac{1}{n(n}}$$

i.e.

$$\sum_{i\neq j}\sigma_i\sigma_j \geq n(n-1)\left(\prod_{i=1}^{n}\lambda_i^{n-1}\right)^{\frac{1}{n(n-1)}} = n(n-1)\left(\prod_{i=1}^{n}\lambda_i\right)^{\frac{1}{n}}$$

Thus $$\sum_{i\neq j}\sigma_i\sigma_j \geq n(n-1)\Delta^{\frac{1}{n}}$$

Where $\Delta = \prod_{i=1}^{n}\lambda_i = \det(A_{mc}A'_{mc})$

Therefore we get

$$[E_{mc}(G)]^2 \geq \sum_{i=1}^{n}\lambda_i + n(n-1)\Delta^{\frac{1}{n}}$$

By Theorem 1 we obtain

$$[E_{mc}(G)]^2 \geq 2\sum_{e\in X}\left(1^2+2^2+...+k_e{}^2\right) + |C| + n(n-1)\Delta^{}$$

Lemma:1, [10]. If $A=[a_{ij}]$ is any non-constant matrix and its norm defined as $\|A\|_2 = \sqrt{\sum_{ij}a_{ij}^2}$. Suppose $\sigma_1 \geq \sigma_2 \geq .... \geq \sigma_n$ are singular values of *A*, then

$$E(A) \geq \sigma_1 + \frac{\|A\|_2^2 - \sigma_1^2}{\sigma_2}.$$

Thus, we evaluate a lower bound for $E_{mc}(G)$

Theorem 6. For a semigraph *G* on *n* vertices, if σ_1 and σ_2 are respectively largest and second largest singular values of its minimum covering matrix $A_{mc}(G)$. Then we have

$$E_{mc}(G) \geq \sigma_1 + \frac{2\sum_{e\in X}\left(1^2+2^2+...+k_e{}^2\right) + |C| - \sigma_1^2}{\sigma_2}$$

Proof: By **Lemma 1**, for the minimum covering matrix $A_{mc}(G)$ of *G*, we have

$$E_{mc}(G) \geq \sigma_1 + \frac{\|A_{mc}\|_2^2 - \sigma_1^2}{\sigma_2}$$

Clearly, from the definition of the norm of a matrix we have

$$\|A_{mc}(G)\|_2^2 = trace(A_{mc}(G)A'_{mc}(G))$$

$$= 2\sum_{e\in X}\left(1^2+2^2+...+k_e{}^2\right) + |C|$$

Hence, we get

$$E_{mc}(G) \geq \sigma_1 + \frac{2\sum_{e\in X}\left(1^2+2^2+...+k_e{}^2\right) + |C| - \sigma_1^2}{\sigma_2}$$

Which gives another lower bound of $E_{mc}(G)$.

V. Relation between minimum covering energy and energy of semigraph

Theorem 7 Let $G(V,X)$ be a semigraph of order *n*, size *m* then

$$E_{mc}(G) \geq \frac{E(G)}{\sqrt{n}}$$

Where $E(G)$ is the energy of semigraph *G*.

Proof: If $G(V,X)$ be a semigraph of order *n*, size *m*, Then by Theorem 2 of [7], we have

$$\sqrt{2\sum_{e\in X}\left(1^2+2^2+...+k_e{}^2\right)} \leq E(G) \leq \sqrt{2n\sum_{e\in X}\left(1^2+2^2+...+K_e^2\right)}$$

$$2\sum_{e\in X}\left(1^2+2^2+....+k_e^2\right) \leq [E(G)]^2 \leq 2n\sum_{e\in X}\left(1^2+2^2+...+k^2\right)$$

Thus

$$[E(G)]^2 \leq 2n\sum_{e\in X}\left(1^2+2^2+...+k^2\right)$$

Therefore

$$\frac{[E(G)]^2}{n} \leq 2\sum_{e\in X}\left(1^2+2^2+...+k^2\right)$$

If $E_{mc}(G)$ be the minimum covering energy of a semigraph $G(V,X)$, by **Theorem 5** we get

$$[E_{mc}(G)]^2 \geq 2\sum_{e\in X}\left(1^2+2^2+...+k_e{}^2\right) + |C| + n(n-1)\Delta^{1/n}$$

i.e $$[E_{mc}(G)]^2 \geq 2\sum_{e\in X}\left(1^2+2^2+...+k_e{}^2\right)$$

Thus $$[E_{mc}(G)]^2 \geq \frac{[E(G)]^2}{n}$$

Hence $$E_{mc}(G) \geq \frac{E(G)}{\sqrt{n}}$$

Theorem 8 For a semigraph $G(V,X)$ of order *n*, size *m*, if σ_1 and σ_2 are respectively largest and second largest singular values of its minimum covering matrix $A_{mc}(G)$. Then we have

$$nE_{mc}(G) \geq \frac{[E(G)]^2 - n\sigma_1^2}{\sigma_2}$$

Where $E(G)$ is the energy of the semigraph.

Proof : If $G(V,X)$ be a semigraph of order *n*, size *m*, Then by Theorem 2 of [7].

$$\sqrt{2\sum_{e\in X}\left(1^2+2^2+...+k_e{}^2\right)} \leq E(G) \leq \sqrt{2n\sum_{e\in X}\left(1^2+2^2+...+K_e^2\right)}$$

Thus

$$[E(G)]^2 \leq 2n\sum_{e\in X}\left(1^2+2^2+....+k_e^2\right)$$

By Theorem 6 we have,

$$E_{mc}(G) \geq \sigma_1 + \frac{2\sum_{e\in X}\left(1^2+2^2+...+k_e{}^2\right)+|C|-\sigma_1^2}{\sigma_2}$$

Thus

$$\sigma_2 E_{mc}(G) - \sigma_1\sigma_2 + \sigma_1^2 \geq 2\sum_{e\in X}\left(1^2+2^2+....+k_e^2\right)+|C|$$

i.e.

$$\sigma_2 E_{mc}(G) - \sigma_1\sigma_2 + \sigma_1^2 \geq 2\sum_{e\in X}\left(1^2+2^2+....+k_e^2\right)$$

i.e.

$$n\left(\sigma_2 E_{mc}(G) - \sigma_1\sigma_2 + \sigma_1^2\right) \geq 2n\sum_{e\in X}\left(1^2+2^2+....+k_e^2\right)$$

i.e. $$n\left(\sigma_2 E_{mc}(G) - \sigma_1\sigma_2 + \sigma_1^2\right) \geq [E(G)]^2$$

VI. Conclusion and Future Work

As it is evident from recent literature, the field of graph energy is a relatively new and rapidly developed area of research that studies various aspects of energy associated with many graph-based models in chemical graph theory. Thus, the study of the energy in the Semigraph model is a very fascinating and challenging area of research that holds great promise for addressing a wide range of real-world problems in the near future.

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Conflict of Interest

The authors have no conflict of interest to declare that is relevant to the content of this article.