Properties of homomorphism and quotient implication algebra on implication algebras

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Abstract: The concept of homomorphisms on implication algebra is introduced. The notion of sub algebras, normal subalgebras in an implication algebra are investigated. Quotient implication algebras and kernels in an implication algebra ,and Homomorphisms and isomorphism theorems are elaborated.

Key-Words: Implication algebra, B-Algebra, Homomorphisms, subalgebra, normal sub algebra and quotient implication algebra.

I. INTRODUCTION

In the study of the properties of a post algebra, Epstein and Horn in [2] introduced the concept of a B-algebra as a bounded distributed lattice with center B in which , for any $a,b \in A$ the largest element $a \Rightarrow b \in B$ exists with the property $a \wedge (a \Rightarrow b) \leq b$. The concept of B-Almost Distributive Lattice (B-ADL) as an ADL in which the lattice of all principal ideals of A is a B-algebra which is initiated by G.C.Rao ; Berhanu,and et at in[3] investigated the concepts of fuzzy congruence relations, and quotient isomorphisms in almost distributive fuzzy lattice, and Naveen Kumar Kakuman in [7] and Joemar in [6] discussed the idea of homomorphism of BF- algebras. Xu in [9] proposed the concept of Lattice Implication Algebras, and discussed their properties; Roh and et al in [8] investigated some operation on lattice implication algebras and Abbott in [1] introduced orthoimplication algebras. Gerima Tefera D. in [4] initiated the idea of Hilbert implication algebra and somproperties and also Gerima T.D in [5] introduced the concept of Subalgebras , Normal subalgebras in an implication algebra , Yang Xu and et al in [10] discussed basic properties and structure of general congruence relations on lattice implication algebra, and Young Bae June in [11] initiated the idea of fuzzy positive implication and fuzzy associative filters of lattice implication algebras. In this paper the concept of Homomorphisms in implication algebras has been introduced. Throughout this paper "$\Rightarrow$" used as a binary operation not as a logical connectives.

II. PRELIMINARIES

Definition 2.1. [6] Let A be a distributive lattice with 0,1 and B , the birkhoff center of A. If for

1. $a \Rightarrow (a \Rightarrow b) = a \Rightarrow b.$

2. *If* $a \in A,$ then $a \Rightarrow (b \Rightarrow c) = (a \wedge b) \Rightarrow c.$

3. *If* $a, b \in A$,the $a \Rightarrow (b \Rightarrow c) = b \Rightarrow (a \Rightarrow c).$

Definition 2.4.[3] An algebra $(A, \Rightarrow, 1)$ of type (2,0) is called implication algebra if the following condition holds:

1. $a \Rightarrow a = 1,$ for all $a \in A$.

2. $a \Rightarrow 1 = 1,$ for all $a \in A.$

3. $1 \Rightarrow a = a,$ for all $a \in A.$

4. $a \Rightarrow (b \Rightarrow c) = b \Rightarrow (a \Rightarrow c),$ forall $a, b, c \in A.$

Definition 2.5. [3] Let A be an implication algebra. Define a relation "$\le$" on A by $a \le b$ if and only if $a \Rightarrow b = 1.$

any $a, b \in A$, there exists a greatest element $y \in A$ such that $a \wedge y \le b$, then A is called a B-algebra.

Proposition 2.2.[6] Let A be a B-ADL, for any $a, b \in A$. Then the following holds:

1.$0 \Rightarrow a = m$ for all $a \in A.$

2.$a \Rightarrow a = m$, for all $a \in A.$

3. $a \Rightarrow m = m$,for all $a \in A,$ m is maximal.

Theorem 2.3.[7] Let A be a B-ADL and $a, b, c \in A.$ Then the following holds :

III. PROPERTIES OF HOMOMORPHISM ON IMPLICATION ALGEBRA

Definition 3.1. Let $(A, \Rightarrow, 1_A)$ and $(B, \Rightarrow, 1_B)$ be implication algebras. Then a mapping $\varphi : A \to B$ is called a homomorphism in an implication algebra if $\varphi(a \Rightarrow_A b) = \varphi(a) \Rightarrow_B \varphi(b)$,

$\forall$ a,b $\in$ A.

Definition 3.2. Let $(A, \Rightarrow, 1_A)$ and $(B, \Rightarrow, 1_B)$ implication algebras. Then ahomomorphism in an implication algebra $\varphi : A \to B$ is called isomorphism if φ is a bijection. If for each b ∈ B, there exists a ∈ A such that φ(a ⇒ c) = b ∈ B. Then φ is called onto homomorphism.

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If for each a,b,c ∈ A, we have φ(a ⇒ c) = φ(b ⇒ d) implies a ⇒ c = b ⇒ d hold, then φ is monomorphism.

Definition 3.3. Let (A,⇒, 1) be an implication algebra. Then a non-empty subset S of A is called a subalgebra of A if a ⇒ b ∈ S , for any a,b ∈ S.

Example 3.1. Let A = {1,a,b,c.d} be a set defined by the table 1 above :

⇒	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	c	d
b	1	1	1	c	c
c	1	a	b	1	b
d	1	1	a	1	1

Table 1. Implication Algebra

Then (A,⇒, 1) is an implication algebra. Here S_1 = {a, 1} and

S_2 = {1,a,b} are subalgebras of A.

Theorem 3.4. Let (A,⇒, 1) be an implication algebra and ; $\phi \neq S \subseteq A$. Then the following are equvivalent:

1. S is a subalgebra of A.

2. a ⇒ (1 ⇒ b), 1 ⇒ b ∈ S, for any a,b ∈ S.

Proof. Let (A,⇒, 1) be an implication algebra and S be non-empty subset of A. Assume S is a subalgebra of A and let a,b, 1 ∈ S. Then a ⇒ (1 ⇒ b) = a ⇒ b,

Since 1 ⇒ b = b We have a ⇒ b ∈ S Since S is a subalgebra of A , and 1 ⇒ b = b, by definition of impliction algebra. Hence 1 ⇒ b ∈ S. Therefore 2 holds.

Assume 2 holds. That is for any a,b ∈ S.a ⇒ (1 ⇒ b) ∈ S and 1 ⇒ b ∈ S. Since a ⇒ b = a ⇒ (1 ⇒ (1 ⇒ b)) = a ⇒ (1 ⇒ b) ∈ S.

Hence a ⇒ b ∈ S, for any a,b ∈ S.

Thus S is asubalgebra of A.

Definition 3.5. Let A be an implication algebra and let ; $\phi \neq N \subseteq A$. Then N is said to be normal in A if (x ⇒ a) ⇒ (y ⇒ b) ∈ N, for any

x ⇒ y, a ⇒ b ∈ N.

Example 3.2. Let A = {0, 1, 2, 3} with 4 as greatset element defined by the table 2 below:

⇒	0	1	2	3	4
0	4	1	2	3	4
1	0	4	2	3	4
2	0	1	4	3	4
3	0	1	2	4	4
4	0	1	2	3	4

Table 2. On normal Ideal

Then (A,⇒, 0, 4) is an implication algebra. Let N = {0, 4} is normal in A.

Since (0 ⇒ 4) ⇒ (4 ⇒ 0) = 4 ⇒ 0 = 0 ∈ N and (4 ⇒ 0) ⇒ (0 ⇒ 4) = 0 ⇒ 4 = 4 ∈ N.

Theorem 3.6. Every Normal subset N of an implication algebra A is a subalgebra of A.

Remark 3.3. The converse of theorem 3.6 doesnot hold. As in example 3.2 N = {1,a} is a subalgebra of A but it is not normal as a ⇒ a,

d ⇒ b ∈ N.

While (d ⇒ a) ⇒ (a ⇒ d) = 1 ⇒ d = d ∉ N.

Lemma 3.4. Let N be a normal subalgebra of an implication algebra A and let a,b ∈ N . Then a ⇒ b ∈ N imply that b ⇒ a ∈ N.

Proof. Let N be a normal subalgebra of an implication algebra A , and let a,b ∈ N with

a ⇒ b ∈ N. Since a ⇒ a = 1 ∈ N and N is normal

b ⇒ a = (a ⇒ a) ⇒ (b ⇒ a) = (a ⇒ b) ⇒ (a ⇒ a) ∈ N, Since a ⇒ a,a ⇒ b ∈ N. Hence b ⇒ a ∈ N.

A. Quotient Implication Algebras

Lemma 3.5. Let (A,⇒, 1) be an implication algebra and let N be a normal sub algebra of A. Define the relation ∼ N on A by a ∼ N bif and only if a ⇒ b ∈ N, where a,b ∈ A . Then ∼ N is an equivalence relation on A.

Proof.

1. Let A be an implication algebra. Then for a,b,c ∈ A, we have 3.1. hold.

Since a ∼ N a ⇔ a ⇒ a = 1 ∈ N.

Hence ∼ N is reflexive.

2. Let a ∼ N b and a,b ∈ N. Then

a ∼ N b ⇔ a ⇒ b ∈ N. b ⇒ a =

(a ⇒ a) ⇒(b ⇒ a) = (a ⇒ b) ⇒ (a ⇒ a) ∈ N,

a ⇒ a = 1,a ⇒ b ∈ N.

Hence b ⇒ a ∈ N.So that b ∼ N a.

Therefore ∼ N is symmetric.

1. Let a,b,c ∈ N and let a ∼ N b and b ∼ N c. Then a ⇒ b ∈ N and

b ⇒ c ∈ N. a ⇒ c = (b ⇒ b) ⇒ (a ⇒ c) =

(b ⇒ a) ⇒ (b ⇒ c) ∈ N.

Hence a ∼ N c. Therefore ∼ N is transitive. Thus ∼ N is an equivalence relation.

Remark 3.6. Let (A,⇒, 1) be an implication algebra. We denote the equivalence class containing a by $[a]_N$. That is $[a]_N = \{b \in A | a \sim N\ b\}$ and $A_N = \{[a]_N | a \in A\}$.

Definition 3.7. Let (A,⇒, 1) be an implication algebra and let N be normal subalgebra of an implication algebra A. Then $[a]_N \Rightarrow [b]_N = [a \Rightarrow b]_N$, ∀ a,b ∈ N.

Remark 3.7. Let A be an implication algebra. Then $[1]_N = \{a \in A | a \sim N1\} = \{a \in A | a \Rightarrow 1 \in N\} = \{a \in A | a \Rightarrow 1 = 1 \in N\} = \{a \in A | 1 \in N\} = N$.

Theorem 3.8. Let N be a normal subalgebra of an impliction algebra A. Then A_N is an implication algebra. Proof Let (A,⇒, 1) be an implication algebra and N be normal. If we define $[a]_N \Rightarrow [b]_N = [a \Rightarrow b]_N$, then the operation " ⇒ " is well defined, since if a ∼ N p and b ∼ N q, then a ⇒ p,b ⇒ q ∈ N implies (a ⇒ b) ⇒ (p ⇒ q) ∈ N by normality of N. Hence (a ⇒ b) ∼ N(p ⇒ q).

Therefore $[a \Rightarrow b]_N = [p \Rightarrow q]_N$. To show A_N is an implication algebra.

1. $[a]_N \Rightarrow [a]_N = [a \Rightarrow a]_N = [1]_N$.

2. $[a]_N \Rightarrow [1]_N = [a \Rightarrow 1]_N = [1]_N$.

3. $[1]_N \Rightarrow [a]_N = [1 \Rightarrow a]_N = [a]_N$.

4. $[a]_N \Rightarrow [b \Rightarrow c]_N = [a \Rightarrow (b \Rightarrow c)]_N = [b \Rightarrow (a \Rightarrow c)]_N = [b]_N \Rightarrow [a \Rightarrow c]_N$.

Hence (A_N ,⇒, $[1]_N$) is an implication algebra. Thus the implication algebra A_N is called the quotient implication algebra of A by N.

Lemma 3.8. Let (A,⇒, 1) be an implication algebra . Then the following holds:

1. The right cancellation law holds. That is a ⇒ b = c ⇒ b implies a = c.

2. If a ⇒ b = 1, then a = b, for any a,b ∈ A.

3. If 1 ⇒ a = 1 ⇒ b, then a = b for any a,b ∈ A.

Proof .

1. Let (A,⇒, 1) be an implication algebra and let a,b ∈ A. Then a,b,c ∈ A with

a ⇒ b = c ⇒ b holds. a = (1 ⇒ a) = (b ⇒ b) ⇒ (1 ⇒ a) = (1 ⇒ b) ⇒ (a ⇒ b) = (1 ⇒ b) ⇒ (c ⇒ b), Since a ⇒ b = c ⇒ b. = (b ⇒ b) ⇒ (1 ⇒ c) = 1 ⇒ c = c .

2. Let a,b ∈ A and a ⇒ b = 1. Then we have a ⇒ b = 1 = b ⇒ b =a ⇒ b = b ⇒ b implies a = b .

3. Let a,b ∈ A and 1 ⇒ a = 1 ⇒ b. It follows by definition 1 ⇒ a = a and 1 ⇒ b = b. As aresult we get a = b.

Definition 3.9. Let $(A, \Rightarrow_A, 1_A)$ and $(B, \Rightarrow_B, 1_B)$ be implication algebras, and

let φ : A → B be homomorphism in an implication algebra. Then

$\{a,c \in A | \varphi(a \Rightarrow_A c) = 1_B\}$ is called the kernel of φ. Denoted by Kerφ.

Theorem 3.10. Let N be a normal subalgebra of an implication algebra A. Then a mapping

γ : A → A N given by $\gamma(a) = [a]_N$ is a surjective implication homomorphism , and Kerγ = N.

Proof. Let N be a normal subalgebra of an implication algebra A and define $\gamma : A \to A_N$ by $\gamma(a) = [a]_N$. Now, $\gamma(a \Rightarrow b) = [a \Rightarrow b]_N = [a]_N \Rightarrow [b]_N = \gamma(a) \Rightarrow \gamma(b)$.

Hence γ is a homomorphism in an implication algebra. For each $[a]_N \in A_N$, there exists $a \in A$ such that $\gamma(a) = [a]_N$.

Hence γ is an onto homomorphism in an implication algebra. Therefore γ is surjective. $Ker\gamma = \{a \in A|\gamma(a) = [1]_N\} = \{a \in A|a \sim N1\} = \{a \in A|a \Rightarrow 1 \in N\}$

$= \{a \in A|1 \in N\} = \{a \in A|\gamma(a) = N\} = N$.

Hence Kerγ = N. The mapping γ discussed here is called the natural (canonocal) homomorphism in an implication algebra onto A_N .

Theorem 3.11. Let $(A, \Rightarrow A, 1_A)$ and

$(B, \Rightarrow B , 1_B)$ be implication algebras and

let $\varphi : A \Rightarrow B$ be a homomorphism in an implication algebra. Then φ is injective if and only if $Ker\varphi = \{1_A\}$.

Theorem 3.12 . Let $\varphi : A \to B$ be a homomorphism in an implication algebra. Then Kerφ is a normal subalgebra of A.

Proof. Let $(A, \Rightarrow A, 1_A)$ and $(B, \Rightarrow B , 1_B)$ be implication algebras and let $\varphi : A \to B$ be homomorphisms in an implication algebras . Since $\varphi(1_A) = \varphi(a \Rightarrow a) = \varphi(a) \Rightarrow \varphi(a) = 1_B$ $1_A \in Ker\varphi$.

Hence $Ker\varphi \neq \phi$.

Let $a \Rightarrow b, x \Rightarrow y \in Ker\varphi$. Then $\varphi(a \Rightarrow b) = 1_B = \varphi(x \Rightarrow y) \Rightarrow \varphi(a) \Rightarrow \varphi(b) = \varphi(x) \Rightarrow \varphi(y)$. Since φ is a homomorphism. Which implies that

$\varphi(a) \Rightarrow \varphi(b) = 1_B$ and $\varphi(x) \Rightarrow \varphi(y) = 1_B$. Implies that $\varphi(a) = \varphi(b)$ and $\varphi(x) = \varphi(y)$.

Now, $\varphi(x \Rightarrow a) \Rightarrow (y \Rightarrow b)) = \varphi(x \Rightarrow a) \Rightarrow \varphi(y \Rightarrow b)$, Since φ is homomorphism

$=(\varphi(x) \Rightarrow \varphi(a)) \Rightarrow (\varphi(y) \Rightarrow \varphi(b))$

$=(\varphi(x) \Rightarrow \varphi(a)) \Rightarrow (\varphi(x) \Rightarrow \varphi(a)) = 1_B$.

Hence $(x \Rightarrow a) \Rightarrow (y \Rightarrow b) \in Ker\varphi$. Thus Kerφ is a normal subalgebra of A.

Lemma 3.9. Let $\varphi : A \to B$ and $\psi : B \to C$ be homomorphisms in an implication algebra . Then $\psi\circ\varphi : A \to C$ is also homomorphism in an implication algebra.

Proposition 3.10. Let $\varphi : A \to B$ be a homomorphism in an implication algebra with 1_A and 1_B be the greatest element in A and B respectively . Then $\varphi(1_A) = 1_B$.

Corollary 3.11. If $\varphi : A \to B$ is a homomorphism in an implication algebra from A into B, then for all $a \in A$, we have $\varphi(1_A \Rightarrow a) = 1_B \Rightarrow \varphi(a)$.

Lemma 3.12. Let $\varphi : A \to B$ be homomorphism in an implication algebra from A into B. Then the following holds:

1. If N is a subalgebra of A, then φ(N) is a subalgebra of B.

2. If S is a subalgebra of B, then $\varphi^{-1}(S)$ is also a subalgebra of containing Kerφ.

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3. If N is a normal subalgebra of A and φ is one-to- one, then φ(N) is a normal subalgebra.

4. If K is a normal subalgebra of B, then φ^{-1} (K) is a normal subalgebra of A.

Proof. Let φ : A → B be homomorphism in an implication algebra from A into B. 1. Let N be a subalgebra of A. Then a ⇒ b ∈ N, ∀ a,b ∈ N ⊆ A. Then

φ(a ⇒ b) = φ(a) ⇒ φ(b) ∈ φ(N) ⊆ B. Since φ(a) ∈ B ,φ(b) ∈ B and φ is homomorphism. Implies that φ(a ⇒ b) ∈ B.

Hence φ(N) ⊆ B. Therefore φ(N) is a subalgebra of B.

2. Let S ⊆ B be a subalgebra of B and c,d ∈ S. Then c ⇒ d ∈ S.

Since $\varphi^{-1}(1_B) = \varphi^{-1}(c \Rightarrow c) = \varphi^{-1}(c) \Rightarrow \varphi^{-1}(c) = a \Rightarrow a = 1_A$. put φ −1 (c) = a. Hence $\varphi^{-1}(1_B) = 1_A \subseteq$ Kerφ.

Now, $\varphi^{-1}(c \Rightarrow d) = \varphi^{-1}(c) \Rightarrow \varphi^{-1}(d) = a \Rightarrow b \in \varphi^{-1}(S) \subseteq A$, $a = \varphi^{-1}(c)$ and $\varphi^{-1}(d) = b \in \varphi^{-1}(S) \subseteq A$. Hence $\varphi^{-1}(S)$ is a subalgebra of A.

3. Let N be a normal subalgebra of A and φ is one-to -one . Then by 1 φ(N) is a subalgebra of B.

4. Let a,b,c ∈ φ(A). Then there exist x, y, z ∈ A such that φ(a) = x,φ(b) = yφ and

φ(c) = z. If x ⇒ y ∈ φ(N), then φ(a ⇒ b) = φ(a) ⇒ φ(b) ∈ φ(N).

Also φ is one- to -one implies a ⇒ b ∈ N, and since N is normal in A, we have

(c ⇒ a) ⇒ (c ⇒ b) ∈ N.

Thus(z ⇒ x) ⇒ (z ⇒ y) = (φ(c) ⇒ φ(a)) ⇒ (φ(c) ⇒ φ(b))

=(φ(c ⇒ a)) ⇒ (φ(c ⇒ b)) = φ((c ⇒ a) ⇒ (c ⇒ b)) ∈ φ(N).

Therefore ,φ(N) is normal subalgebra of φ(A).

5. Let K be a normal subalgebra of B by 2 φ^{-1} (K) is a subalgebra of A.

Let a,b,c ∈ A. If a ⇒ b ∈ φ^{-1} (K), then φ(a ⇒ b) = φ(a) ⇒ φ(b) ∈ K Since φ is homomorphism.

Since K is normal subalgebra of B and φ(c) ∈ B,φ((c ⇒ a) ⇒ (c ⇒ b)) = φ(c ⇒ a) ⇒ φ(c ⇒ b) Since φ is homomorphism =(φ(c) ⇒ φ(a)) ⇒ (φ(c) ⇒ φ(b)) ∈ K. Since φ is homomorphism in an implication algebra.

Hence (c ⇒ a) ⇒ (c ⇒ b) ∈ φ^{-1} (K). Thus φ^{-1} (K) is a normal subalgebra of A.

Theorem 3.13. Let $(A,\Rightarrow A, 1_A)$ and

$(B,\Rightarrow B , 1_A)$ be implication algebras and

let φ : A → B be a homomorphism from A onto B. Then $A_{K er\varphi}$ is isomorphic to B.

Proof. Let φ : A → B be ahomomorphism in an inplication algebra. We need to show A K erφ ∼= Imφ. Since K erφ is normal subalgebra of A, we have A K erφ is a quotient implication algebra by lemma 3.12 Imφ is an implication algebra.

Let Kerφ = N. Then define $f : A_N \rightarrow$ Imφ by $f([a]_N) = \varphi(a)$. Now, let $[a]_N , [b]_N \in$ A N . Then $f([a]_N) = \varphi(a)$ and $f([b]_N) = \varphi(b)$. So that f ([a

$\Rightarrow b]_N) = \varphi(a \Rightarrow b) = \varphi(a) \Rightarrow \varphi(b) = f([a]_N) \Rightarrow f([b]_N)$. Hence f is ahomomorphism in an implication algebra.

Also set $f([a]_N) = f([b]_N) \Rightarrow \varphi(a) = \varphi(b) \Rightarrow \varphi^{-1}\varphi(a) = \varphi^{-1}\varphi(b)$

$\Rightarrow a = b \Rightarrow [a]_N = [b]_N$.

Thus f is monomorphism in an implication algebra. Moreover , let $b \in Im\varphi$, then there exists $a \in A$ such that $\varphi(a) = b = f([a]_N)$.

Hence f is epimorphism. Thus f is isomorphism in an implication algebra. Consequentely $A_{Ker\varphi} \cong B$.

Theorem 3.14. (Second isomorphism theorem) Let N and K be normal subalgebras of an implication algebra A. Then N N ∩K ∼= NK K.

Proof. Let N and K be normalsubalgebras of an implication algebra A. Then define $\varphi : N \to NK$ K by $\varphi(a) = a$ k . for any $a \in N$.

Let $a, b \in N$. If $a = b$, then

$a \Rightarrow b = a \Rightarrow a = 1_A \in K$. That is $a \sim K$ b. Thus a k = b k . So that $\varphi(a)$ = a k = b k = $\varphi(b)$.

Hence $\varphi(a) = \varphi(b)$. This shows that φ is well defined. Moreover,

$\varphi(a \Rightarrow b) = a \Rightarrow b_k = a_k \Rightarrow_A b_k = \varphi(a) \Rightarrow_A \varphi(b)$.

Therefore φ is homomorphism in an implication algebra. If c k ∈ NK K , then

$c = a \Rightarrow (1 \Rightarrow b)$ for some $a \in N, b \in K$.

So that we get $c_k = a \Rightarrow (1 \Rightarrow b)_k = a_k \Rightarrow_A b_k = \varphi(c)$. Hence φ is onto.

Thus by theorem 3.13. $N_{Ker\varphi}$ ∼= nk k . Furthermore,$Ker\varphi = \{a \in N : \varphi(a) = (1 \Rightarrow (1 \Rightarrow 1)/k = 1_K\} = \{a \in N : a_k = 1_K\} = \{a \in N : a \sim K1\} = \{a \in N : a = 1 \Rightarrow a \in K\} = N \cap K$. Therefore N K ∩ N ∼= NK K .

Lemma 3.13. If N and K are normal subalgebras of an implication algebra A such that $N \subseteq K$. Then K N is normal subalgebra of A N .

Proof. Let $(A, \Rightarrow A, 1_A)$ be an implication algebra and let N and K be normalsubalgebras of an implication algebra A such that $N \subseteq K$. Then K N ⊆ A N . Now, 1 N ∈ K N Since 1 ∈ K. Thus K N is not empty.

If a N b N ∈ K N and b N ∈ K N and a N ∈ K N . Hence $a \Rightarrow b \in K$. Thus $a \Rightarrow b$ N ∈ K N . Therefore K N is a sub algebra.

Again let a N , b N , c N ∈ A N . If a N ⇒A b N ∈ K N , then a ⇒ b N ∈ K N and a ⇒ b N = a N ⇒A b N ∈ K N . Hence $a \Rightarrow b \in K$. Since K is normal subalgebra in A ,

$(c \Rightarrow a) \Rightarrow (c \Rightarrow b) \in K$. Thus $(c \Rightarrow a) \Rightarrow (c \Rightarrow b)$ N ∈ K N .

So that (c N ⇒A a N) ⇒A (c N ⇒A b N) = (c ⇒ a) N ⇒A (c ⇒ b) N = ((c ⇒ a)) ⇒ (c ⇒ b)) N ∈ K N .

Therefore , K N is normal subalgebra in A N .

Theorem 3.15 (Third isomorphism theorem). If N and K are normal subalgebras of an implication algebra A such that $N \subseteq K$, then (A/N) (K/N) ∼= A K.

Proof . Let A be an implication algebra and let N and K be normal subalgebra of an implication algebra A such that N ⊆ K. Then define the function $\varphi : A_N \rightarrow A_K$ by $\varphi(a_N) = a_K$, $\forall a_N \in A_N$. Let $a_N, b_N \in A_N$. If $a_N = a_N$, then a ∼ N b. That is a ⇒ b ∈ N ⊆ K. Thus a ∼ K b and a K = b K . Hence $\varphi(a_N)$ = a K = b K = φ(b N). Therefore φ is well defined. Moreover,φ is ahomomorphism , since $\varphi(a_N \Rightarrow_A b_N)$ = φ(a⇒b N) = a ⇒ b K = a K ⇒A b K = $\varphi(a_N) \Rightarrow_A \varphi(b_N)$. If a K ∈ A K , then a N ∈ A N since N ⊆ K, and $\varphi(a_N)$ = a K . Thus φ is onto. By theorem 3.13 (A/N) K erφ ∼= A K . K erφ = { a N ∈ A N : φ(a N) = 1K } = { a N ∈ A N : a K = 1K }

= { a N ∈ A N : a ∼ K1} = { a N ∈ A N : a = 1 ⇒ a ∈ K} = K N .

Therefore (A/N) (K/N) ∼= A K .

IV. CONCLUSION

The concepts of sub algebras, normal subalgebras in an implication algebra have been introduced. The kernel and image of homomorphism in an implication have been characterized. In addition the homomorphisim in an implication algebra has been introduced and basic homomorphism theorems like first isomorphism theorem, second isomorphism theorem, and third isomorphism theorems have been discussed in an implication algebra with their proofs.

Data Availability

The data used to support the findings of this study are included by citation with in the study of the article. I allow this manuscript to be available as open access for readers and researchers. No figures, photo and pictures in main manuscript and no separate tables.

Conflicts of Interest

There is no conflict of interest between authors.

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