Ranks of Identity Difference Transformation Semigroup

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Abstract-This study focuses on the ranks of identity difference transformation semigroup. The ideals of all the (sub) semigroups; identity difference full transformation semigroup (IDT_n), identity difference order preserving transformation semigroup, (IDO_n), identity difference symmetric inverse transformation semigroup(IDI_n), identity difference partial order preserving symmetric inverse transformation semigroup($IDPOI_n$) and identity difference partial order preserving transformation semigroup ($IDPO_n$) were investigated for rank and their combinatorial results obtained respectively.

Keywords: Transformation Semigroup, Identity Difference, Rank.

I. Introduction

The rank of transformation semigroup on a set X_n has been widely studied. Amongst the earliest studies, Gomes and Howie obtained the rank and idempotent rank of partial order preserving transformation semigroups PO_n to be $(2n-1)$ and $(3n-2)$ and that of O_n to be respectively (n) and $2(n-1)$. Also they showed in the case of 'strictly partial order preserving maps' $SPO_n = PO_n \backslash O_n$ while rank of SPO_n is $2(n-1)$ for all $n \geq 2$.

A generalization of this study initiated by Garba in 1994 showed that the semigroups $L(n,r), M(n,r)$ and $N(n,r)$ have equal rank with idempotent rank. That is, $L(n,r) = \binom{n}{r}, M(n,r) = \sum_{k=r}^{n}\binom{n}{k}\binom{k-1}{r-1}$ and $N(n,r) = \sum_{k=r}^{n-1}\binom{n}{k}\binom{k-1}{r-1}$ respectively.

The relative rank $(S:A)$ of a subset A of a semigroup S is the minimum cardinality of a set B such that, $\langle A \cup B\rangle = S$ The monoid of all contraction in T_n is of uncountable relative rank. [9]

The term compression map in partial order-preserving transformation semigroup was used by Zhao and Yang in 2012 to be $CPO_n = \{\alpha \in\in PO_n : (\forall\ x,y \in dom\alpha), |x\alpha - y\alpha| \leq |x-y|$ where they characterized the Green's relation on CPO_n and the regularity of the elements of CPO_n. [13]

The definition of compression maps as in [13] is the same as contraction maps in [9],[6],[3], etc. The study in [13] pave way for further characterization of the Green's relation on the subsemigroups CT_n and OCT_n, which gave birth also to the characterization of both CT_n and OCT_n on the Green's starred relations [6]. The cardinalities of these subsemigroups OCT_n and $ORCT_n$ were investigation and $|OCT_n| = (n+1)2^{n-2}$ and $|ORCT_n| = (n+1)2^{n-1} - n$ were obtained [3].

Symons in [15] denoted the full transformation semigroup to be $T(X)$ and studied the automorphism and isomorphism of $T(X,Y) = \{\rho \in T(X): X\rho \subseteq Y\}$ and as such $T(X,Y) = T(X)$ if $Y = X$ with Y a fixed nonempty subset of X. In [12] they characterized the regular elements of $T(X,Y)$, amongst other results obtained from the set $F(X,Y) = \{\rho \in T(X,Y): X\rho = Y\rho\}$ which consists of regular elements in $T(X,Y)$. Sanwong and Sommanee [14] characterized the Green's relation on the subsemigroup $T(X,Y)$ and proved that $F(X,Y)$is the largest regular subsemigroup of $T(X,Y)$. Also, they obtained the rank and idempotent rank of $F(X,Y)$ ideals with X a finite set. The subsemigroups $P(X,Y)$and $I(X,Y)$ of the partial transformation semigroup $P(X)$ was defined in [5] as follows; $P(X,Y) = \{\rho \in P(X): im\rho \subseteq Y\}$ and $I(X,Y) = \{\rho \in I(X): im\rho \subseteq Y\}$ and further obtained that $PF(X,Y) = \{\rho \in P(X,Y): im\rho = Y\rho\}$ is the largest regular subsemigroup of $P(X,Y)$, and $I(Y)$ is the largest inverse subsemigroup of $I(X,Y)$ [5].

In [1] the identity difference transformation semigroups were introduced by using the formulae $\max(im(\propto) - \min(im(\propto) \leq 1$ and the combinatorial results for $|IDT_n|$, $|IDI_n|$, $|IDP_n|$, $|OIDT_n|$, $|OIDI_n|$, and $|OIDP_n|$ was obtained respectively. More combinatorial results for the nilpotents, $|N_n|$, idempotents, $|E(S)|$, and fix of the subsemigroups of the transformation semigroups was also obtained.

In [2] it was shown that $IDT_n$is a subsemigroup of the full transformation semigroup T_n and congruence property of the Green's relations $\mathcal{L}$ and $\mathcal{R}$ examined on IDT_n.

In [11] the study of the semigroup $\overline{S}$ yield some combinatorial results for $\bar{S}$, $W(\bar{S})$ and $\overline{W}(\bar{S})$ For all $n \geq 3, |\bar{S}| = \bar{S} = 3 + \sum_{i=0}^{n}(i-1) + (1-i)$ $\mathcal{W}(\bar{S}) = n^2 - 3n + 2$

$\overline{\mathcal{W}}(\bar{S}) = \frac{n^2-3n+2}{3}$ average work done.

This study investigates the identity difference transformation semigroup for ranks for $n \geq 3$.

where Section 2 of this work deals with the definition of terms, section 3 shows the properties of the ranks of the identity difference transformation semigroup, section 4 provides the main results and conclusion of the study respectively.

II. DEFINITION OF TERMS

A transformation is a map from a set X to itself (self-map). That is $f: X \to X$.

The partial transformation semigroup $\mathcal{PT}_n$ is the semigroup of all partial transformations on the finite set $N = \{1,2,\dots,n\}$ with respect to the composition of map.

The Full transformation semigroup $\mathcal{T}_n$ on a set n is the semigroup of all transformations on X (that is, all mappings from X to itself) under the operation of composition of map.

The Symmetric group S_n on a set X is the group consisting of all bijection from the set X, where $|X| = X$ to itself with function composition as the group operation. Note that $S_n = \mathcal{T}_n \cap \mathcal{I}_n$.

The set of all partial bijection(s) on a set X, where $|X| = X$ (that is, one-to-one partial transformation) forms an inverse semigroup called the symmetric inverse semigroup $\mathcal{I}_n$.

A binary relation $\leq_X$ on a set X is said to be a (partial) order if

I $x \leq_X x \;\forall\; x \leq_X$. ($\leq_X$ is reflexive).

II $x \leq_X y$ and $y \leq_X x$, then

$x = y \;\forall\, x, y \in \leq_X$. ($\leq_X$ is antisymmetric).

III $x \leq_X y$ and $y \leq_X z$ then $x \leq_X z \;\forall\, x, y, z \in \leq_X$. ($\leq_X$ is transitive).

Let $(A, \leq_A)$ and $(B, \leq_B)$ be partially ordered sets. The map $\alpha: A \to B$ is order preserving if $a \leq_A a'$ implies $\alpha(a) \leq_B \alpha(a')$ ($\forall\, a, a' \in A$).

A mapping $\alpha \in \mathcal{T}_n$ is said to be order preserving if for any $x, y \in N$, then $x \leq y \Rightarrow \alpha(x) \leq \alpha(y)$.

The set of all order preserving maps forms a semigroup and is denoted by $\mathcal{O}_n$.

Let $\alpha : W \to X$ and $\beta : X \to Y$ then the composition of these functions

$\alpha * \beta : W \to Y$, can be defined as $x(\alpha * \beta) = x(\alpha)\beta$ ($\forall x \in W$).

Also, the composition of these functions is associative if there exist another function $\gamma: Y \to Z$, then; $x\big((\alpha * \beta) * \gamma\big) = \big(x(\alpha * \beta)\big)\gamma = \big((x\alpha)\beta\big)\gamma = (x\alpha)(\beta * \gamma) = x\big(\alpha * (\beta * \gamma)\big)$ ($\forall x \in W$).

The term identity difference is defined by the rule

$W^{+}(\alpha) - W^{-}(\alpha) \leq 1$ or $\mathrm{Max}(im\alpha) - \mathrm{Min}(im\alpha) \leq 1$.

The map $\alpha \in PT_n$ is said to be contraction if $|\alpha(a) - \alpha(b)| \leq |a - b| \;\forall\, a, b \in X_n$.

Let $S = IDPT_n$, the $Rank(S) = Min\{|A| : A \subseteq S \text{ and } < A > = S\}$.

For more definitions see [10], [18], [40], [41], etc.

III. PROPERTIES OF THE RANK OF IDENTITY DIFFERENCE TRANSFORMATION SEMIGROUP

In this section we shall investigate the semigroups IDT_n, IDO_n, IDI_n, $IDPOI_n$ and $IDPO_n$ for rank and summarize their properties using $\mathcal{R}$ classes of each of the semigroups under consideration.

For simplicity's sake we shall arrange the element of the said semigroup in their respective $\mathcal{R}$ classes.

A. IDENTITY DIFFERENCE FULL TRANSFORMATION SEMIGROUP IDT_n

Let the element $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$ be represented by (112) and $IDT_n = S$

Consider the $\mathcal{R}$ classes of S to be R_1, R_2, R_3 and R_4 for $n = 3$ where

$R_1 = \{(112), (221), (223), (332)\}$,

$R_2 = \{(121), (212), (232), (323)\}$,

$R_3 = \{(122), (211), (233), (322)\}$ and

$R_4 = \{(111), (222), (333)\}$.

In S, R_4 Contains the elements with the constant maps ς.

The minimum generating set for S is

$A = \{(112), (233), (323)\}$. By observation the constant map $\varsigma \notin A$. Hence $Rank(S) = 3 \;\forall\, n = 3$. Using the same approach above, the following sequence were obtained for a general case. 3,7,15,31,63, . . .$\forall\, n \geq 3$. Basically, the rank of the semigroup S is observed to be $(\mathcal{R} - 1)$ where ''1'' is the number of set containing the constant maps.

The above illustration is same for identity difference order-preserving transformation semigroup IDO_n.

B. IDENTITY DIFFERENCE SYMMETRIC INVERSE TRANSFORMATION SEMIGROUP

let $\boldsymbol{S = IDI_n}$.

Consider the $\mathcal{R}$ classes (R_1, R_2, R_3 and R_7)of the semigroup S for $n = 3$

$R_1 = \{(12-), (21-), (23-), (32-)\}$,

$R_2 = \{(1-2), (2-1), (2-3), (3-2)\}$,

$R_3 = \{(-12), (-21), (-23), (-32)\}$ and

$R_4 = \{(1--), (2--), (3--),$

$R_5 = \{(-1-), (-2-), (-3-), \}$,

$R_6 = \{(--1), (--2), (--3)\}, R_7 = \{(---)\}$

are the elements of IDI_n in their respective $\mathcal{R}$ classes. R_4, R_5, R_6 and R_7 Contains the elements with the identity, and empty maps. After some investigations we observe that the generating set can be obtained by picking an element randomly from each of the $\mathcal{R}$ classes (R_1, R_2 and R_3). Hence A contains 3 elements for $n \geq 3$. In general, the minimum generating set A for S is 3,6,10,15, . . .$\forall\, n \geq 3$.

The illustration for IDI_n is same for $IDPOI_n$ and they also have same sequence of minimum generating sets. Hence, IDI_n and $IDPOI_n$ has equal ranks.

C. IDENTITY DIFFERENCE PARTIAL ORDER SYMETRIC INVERSE TRANSFORMATION SEMIGROUP $IDPOI_n$

Consider the $\mathcal{R}$ classes (R_1, R_2, R_3 and R_7)of the semigroup S for $n = 3$

$R_1 = \{(12-), (23-), \}$,

$R_2 = \{(1-2),\ (2-3), \}$,

$R_3 = \{(-12),\ (-23), \}$ and

$R_4 = \{(1--), (2--), (3--)$,

$R_5 = \{(-1-), (-2-), (-3-), \}$,

$R_6 = \{(--1), (--2), (--3)\}, R_7 = \{(---)\}$

D. IDENTITY DIFFERENCE PARTIAL-ORDER PRESERVING TRANSFORMATION SEMIGROUP $IDPO_n$

$R_1 = \{(112), (223)\},\ R_2 = \{(122), (233)\}$,

$R_3 = \ \{(12-), (23-)\}, R_4 = \{(-12), (-23)\}$,

$R_5 = \{(1-2), (2-3)\}, R_6 = \{(111), (222), (333)\}$,

$R_7 = \{(11-), (22-), (33-)\}$,

$R_8 = \ \{(-11), (-22), (-33)\}$,

$R_9 = \{(1-1), (2-2), (3-3)\}$,

$R_{10} = \ \{(1--), (2--), (3--)\}$,

$R_{11} = \{(-1-), (-2-), (-3-)\}$,

$R_{12} = \{(--1), (--2), (--3)\}\ R_{13} = \{(---)\}$

are the elements of $IDPO_3$ in their respective $\mathcal{R}$ classes. $R_6, R_7,\ .\ .\ .\ R_{13}$ Contains the elements with the identity, constant and empty maps. After some mathematical operations we obtain the generating set by picking an element randomly from each $\mathcal{R}$ classes ($R_1,\ R_2, .\ .\ .\ R_5$). Generally, the minimum generating set A for S is of the sequence $5,17,49,129,\ .\ .\ .\ \forall\, n \geq 3,\ R_1, R_2, R_3,\ R_4,\ .\ .\ .\ R_{13} \in \mathcal{R}$.

IV. THE RANKS OF IDENTITY DIFFERENCE TRANSFORMATION SEMIGROUP

In this section we shall give the theoretical and combinatorial proves for the ranks of IDT_n, IDO_n, IDI_n, $IDPOI_n$ and $IDPO_n$ respectively.

A. RANK OF IDENTITY DIFFERENCE FULL TRANSFORMATION SEMIGROUP IDT_n

THEOREM 1

Let $\alpha \in S$ and $R(S)$ denote the rank of S. $\alpha \in R(S)$ iff $\alpha \in A$ is the generating set of S.

Proof

Let S be the identity difference transformation semigroup. Suppose $(\alpha, \beta), (a, b), (c, d),\ .\ .\ . \in \mathcal{R}$ of S. Then $(\alpha, b, d,\ .\ .\ .) \in A$ if$(\alpha, b, d,\ .\ .\ .)\backslash \varsigma$, (where ς is a constant map elements). Suppose also that the choice of picking $\alpha, b, d,\ .\ .\ .$ in $(\alpha, \beta), (a, b), (c, d),\ .\ .\ .$ of S is done randomly, then the products $(\alpha. b), (b. \alpha), (b. \alpha)d,\ .\ .\ .$ must generate the set of elements in S. Hence $\{\alpha, b, d, \ldots) \in A$ is the generating set for S since one element is selected at random from each $\mathcal{R}$ which excludes the $\mathcal{R}$ classes containing the constant or identity maps.

In contradiction, Since $(\alpha, \beta), (a, b), (c, d),\ .\ .\ . \in \mathcal{R}$ of S. If the choice of selecting the elements of A from different sets of $\mathcal{R}$ are random and (α, β, a, b) are selected for A, then $A \nsubseteq S$. Hence, A is not a generating set for S since the choice of elements does not consider the above stated conditions.

Thus, $\alpha \in R(S)$ iff $\alpha \in A$ and $A \subseteq S$. □

THEOREM 2

Let$S = IDT_n$. If $|\mathcal{R}|$ be the order of $\mathcal{R}$ in S. Then Rank (S) is $(|\mathcal{R}| - 1) = 2^{(n-1)} - 1$.

Proof

Let $uS, \ldots n + (n+1)(2^n - 2)S$ be the set of right ideals in $S\ \forall\, u \in S$. If $aS = bS = cS = \cdots nS$ ($\forall\, a, b, c, \ldots, n \in S$); then the elements in the set $\{a, b, c, \ldots, n\}$ are in same $\mathcal{R}$.

Suppose $a, b, c, \ldots, n$ are set of idempotent elements, then $|\text{im}\varepsilon|$ =1 where ε is a constant map. Hence $a, b, c, \ldots, n\ \varepsilon\ E(S)$.

Let $uS = vS = \cdots 2(n-1)S$ be set of equal right ideals in S where $\{p, q, \ldots 2(n-1)\}$, $\{m, n, \ldots 2(n-1)\}, \ldots\ |\mathcal{R}| \setminus \mathcal{R}_c$ are different sets of $\mathcal{R}$ in S. Then $p, q, \ldots, 2(n-1)$ (respectively $m, n, \ldots\ 2(n-1)$) are elements in the same $\mathcal{R}$ where $\mathcal{R}_c$ is a set of $\mathcal{R}$ containing the constant map.

Suppose that $|\mathcal{R}|$ in S contain the set $\mathcal{R}_c$ such that $\forall\, \alpha \in S\ \alpha(X_1\ .\ .\ .X_n) = i, i \geq 1$ is a constant map. Then the $Rank(S) = Min\{|A|: A \subseteq S, < A > = S\}$ is $|\mathcal{R}| - |\mathcal{R}_c| = |\mathcal{R}| - 1$.

Without loss of generality, for $n \geq 3$, $(aS = bS = \cdots = nS), (pS = qS = \cdots = 2(n-1), (mS = nS = \cdots = 2(n-1), \ldots 2^{(n-1)}$ implying that there are $2^{(n-1)}$ $\mathcal{R}$ in S.

Hence the $Rank(S) = |\mathcal{R}| - 1 = 2^{(n-1)} - 1 = 3,7,15,31,63, \ldots \forall\, n \geq 3$. □

See the table below;

$n \geq 3$	$Rank = (\lvert\mathcal{R}\rvert - 1) = 2^{(n-1)} - 1)$
3	3
4	7
5	15
6	31
7	63
8	127
9	255
10	511

THEOREM 3

Let S be identity difference full transformation semigroup. The

$$Rank(S) = \frac{1}{2}\sum_{p=0}^{n}\binom{n}{p} - \binom{n}{n} = 2^{(n-1)} - 1$$

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Proof

Let $R(S) = \frac{1}{2}\sum_{P=0}^{n}\binom{n}{p} - \binom{n}{n}$

Implying that there are n ways P (respectively n) can be represented since $\binom{n}{p}$ and $\binom{n}{n}$.

Recall the identity

$\sum_{p=0}^{n}\binom{n}{p} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n-2} + \cdots + \binom{n}{n} = 2^n$. So that,

$\frac{1}{2}\sum_{p=0}^{n}\binom{n}{p} = \sum_{p=0}^{n}\frac{1}{2}\binom{n}{p} = \frac{1}{2}\left[\binom{n}{o} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-2} + \cdots + \binom{n}{n}\right] = \frac{1}{2}[2^n] = 2^{-1}.2^n = 2^{n-1}$

Therefore $\frac{1}{2}\sum_{p=0}^{n}\binom{n}{p} - \binom{n}{n} = 2^{-1}.2^n - 2^0 = 2^{n-1} - 1$.

□

B. RANK OF IDENTITY DIFFERENCE ORDER PRESERVING TRANSFORMATION SEMIGROUP IDO_n

Following the same approach as in 4.1 above, we obtain that $Rank(S) = (n-1) \forall\, n \geq 3$.

Lemma 1

Let S be IDT_n and IDO_n. If $nR_{classes}$ is the number of set of $R - classes$ in S. Then $Rank(S)$ is defined by the rule $(nR_{classes} - 1)$.

THEOREM 4

Let $S = IDO_n$. If $|\mathcal{R}|$ be the order of $\mathcal{R}$ in S. Then Rank (S) is $(|\mathcal{R}| - 1) = (n-1)$ □

Since $Rank(S) = (n-1)\ \forall\, n \geq 3$ then,

$n \geq 3$	$Rank = (n-1)$
3	2
4	3
5	4
6	5
7	6
8	7
9	8
10	9

□

THEOREM 5

$Rank(S) = \binom{n-1}{n-2} = (n-1)$.

Proof

The expression $\binom{n-1}{n-2}$ is true that (n-1) can be represented in (n-2) ways and as such we have,

$\binom{n-1}{n-2} = \frac{(n-1)!}{[(n-1)-(n-2)]!(n-2)!} = \frac{(n-1)!}{(n-2)!\,.1!} = \frac{(n-1)!(n-2)!}{(n-1)!} = (n-1)$

Hence $\binom{n-1}{n-2} = (n-1)$ □

C. RANK OF IDENTITY DIFFERENCE SYMMETRIC INVERSE TRANSFORMATION SEMIGROUP IDI_n

THEOREM 6

Let $IDI_n = S$ be the identity difference symmetric inverse transformation semigroup. If $|\mathcal{R}|$ be the order of the set of R-classes in S then $Rank\ (S) = \big(|\mathcal{R}| - (n+1)\big) = (n-1) + \sum_{i=1}^{n-2}(i)$

Proof

Let $\alpha_1 S, \alpha_2 S, \ldots \alpha_{n+1+n^2(n-1)} S$ be the right ideals of $S\ \forall\ \alpha_1, \alpha_2, \ldots\ \alpha_{(n+1)+n^2(n-1)} \in S\ \forall n \geq 3$.

Where $\alpha_1 = (\begin{pmatrix} x_1 & x_2 \\ i & i+1 \end{pmatrix} \ldots\ x_n), \alpha_2 = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ i+1 & i & \ldots & - \end{pmatrix}, \alpha_3 = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ - & i & \ldots & i+1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ - & i+1 & \ldots & i \end{pmatrix}, \ldots \alpha_{t-1} = \begin{pmatrix} x_1 & \ldots & x_{n-1} & x_n \\ - & \ldots & i & i+1 \end{pmatrix}, \alpha_t = \begin{pmatrix} x_1 & \ldots & x_{n-1} & x_n \\ - & \ldots & i+1 & i \end{pmatrix}$.

Observe that, for all elements from $\alpha_{1,\ldots,}\alpha_t \in S$ there are precisely two point of maps from the domain to the codomain and every other point in each element are empty. As such, $\{\alpha_1, \alpha_2, \ldots\} \in S, \{\alpha_{3,}\alpha_4, \ldots\} \in S, \ldots, \{\alpha_{t-1,}\alpha_t \ldots\} \in S$ since, $\alpha_1 S = \alpha_2 S = \cdots, \alpha_3 S = \alpha_4 S = \cdots, \alpha_{t-1} S = \alpha_t S = \cdots$.

Similarly. Let $\beta_{1,}\beta_{2,} \ldots, \beta_t \epsilon\ S$ such that, $\beta_1 = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ i & - & \ldots & - \end{pmatrix}, \beta_2 = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ i+1 & - & \ldots & - \end{pmatrix} \ldots, \beta_{t-1}\begin{pmatrix} x_1 & \ldots & x_{n-1} & x_n \\ - & \ldots & - & i \end{pmatrix}, \ldots$

$\beta_t\begin{pmatrix} x_1 & \ldots & x_{n-1} & x_n \\ - & \ldots & - & i+1 \end{pmatrix} \ldots$

If $\beta_1 S = \beta_2 S \ldots, \beta_{t-1} S = \beta_t S \ldots, then\ \{\beta_{1,}\beta_2 \ldots\} \epsilon \mathcal{R} \ldots \{\beta_{t-1}, \beta_t \ldots\} \epsilon\ \mathcal{R}\ in\ S.$

Recall that in S, there exist the empty map ξ such that, $\xi(x_{1,}x_2 \ldots x_n) = \xi$, that is $\xi(x_1) = \xi, \xi(x_2) = \xi, \ldots, \xi(x_n) = \xi$. Hence $\{\xi\}$ is a set in $\mathcal{R}$

Therefore, $|\mathcal{R}| = (\{\alpha_1, \alpha_2, \ldots, \alpha_{2(n-1)}\}, \{\alpha_3, \alpha_4, \ldots, \alpha_{2(n-1)}\}, \ldots \{\alpha_{t-1}, \alpha_t, \ldots, \alpha_{2(n-1)}\}$ $, \ldots \{\beta_{1,}\beta_2, \ldots, \beta_n\}, \ldots \{\beta_{t-1}, \beta_{t1} \ldots \beta_n\}, \ldots, \xi$, Since by definition, Rank (S)=$Min\{|A|: A \subseteq S, < A > = S\}$, observe that the generating set is obtained by picking an element from each set of $\mathcal{R}$ (say $\alpha_{1,}\alpha_4, \alpha_6, \alpha_t, \ldots, \alpha_{(n-1)+\sum_{p=1}^{n-2} p} = A$) whose element must contain precisely two points of mappings and every other points empty map. Also, the choice of picking the elements of A is of high importance since $< A > = S$.

Without loss of generality, in $\mathcal{R}(S)$

$|\mathcal{R}_1| =$
$[\{\alpha_1, \alpha_{2,} \dots \alpha_{2(n-1)}\}, \{\alpha_3, \alpha_4, \dots, \alpha_{2(n-1)}\}, \dots \{\alpha_{t-1}, \alpha_t, \dots \alpha_{2(n-1)}\} \dots] =$
$(n-1) + \sum_{i=1}^{n-2}(i)$ cases

$|\mathcal{R}_2| = [\{\beta_1, \beta_2, \dots, \beta_n\}, \{\beta_{t-1}, \beta_t, \dots, \beta_n\}] = n$ number of cases

$|\mathcal{R}_3| = \{\xi\} = 1$ case.

As such the Rank(S)= $[|\mathcal{R}| - (n+1)] = (n-1) + \sum_{i=1}^{n-2}(i)$

See the table below,

$n \geq 3$	$\lvert\mathcal{R}\rvert$	$Rank(S) = (n-1) + \sum_{i=1}^{n-2} i$
3	7	3
4	11	6
5	16	10
6	22	15
7	29	21
8	37	28
9	46	36
10	56	45

THEOREM 7

$Rank(S) = \binom{n-1}{n-2} + \sum_{(i=1)}^{(n-2)} \binom{i}{i-1}$

Proof

$= \frac{(n-1)!}{[(n-1)-(n-2)]!(n-2)} + \sum_{i=1}^{(n-2)} [\frac{i!}{[i-(i-1)]!(i-1)!}] \quad = \frac{(n-1)!}{(n-2)!} + \sum_{i=1}^{n-2} (\frac{i!}{(i-1)!}) = \frac{(n-1)(n-2)!}{(n-2)!} + \sum_{i=1}^{n-2} (\frac{i(i-1)!}{(i-1)!})$

$= (n-1) + \sum_{i=1}^{n-2}(i)$. Therefore, $Rank(S) = (n-1) + \sum_{i=1}^{n-2}(i)$ □

D. RANK OF IDENTITY DIFFERENCE PARTIAL ORDER SYMETRIC INVERSE TRANSFORMATION SEMIGROUP $IDPOI_n$

THEOREM 8

Let $S = IDPOI_n$. Then

$Rank(S) = \binom{n-1}{n-2} + \sum_{(i=1)}^{(n-2)} \binom{i}{i-1} = (n-1) + \sum_{i=1}^{n-2}(i)$

□

THEOREM 9

Let $IDPOI_n = S$. If $|\mathcal{R}_c|$ be the order of the set of R-classes in S then

$Rank\ (S) = (|\mathcal{R}| - (n+1)) = (n-1) + \sum_{i=1}^{n-2}(i)$ (See theorem 6). □

E. RANK OF IDENTITY DIFFERENCE PARTIAL ORDER PRESERVING TRANSFORMATION SEMIGROUP $IDPO_n$.

THEOREM 10

Let $S = IDPO_n$. Then $Rank(S) = \left[\frac{1}{2}\sum_{p=0}^{n}\binom{n}{p}\right]\binom{n-2}{n-3} + \binom{n}{n} = 2^{n-1}(n-2) + 1$

Proof

$Rank(S) = \left[\frac{1}{2}\sum_{p=0}^{n}\binom{n}{p}\right]\binom{n-2}{n-3} + \binom{n}{n} = \frac{1}{2}\left[\binom{n}{o} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-2} + \dots + \binom{n}{n}\right]$
$(\frac{(n-2)!}{[(n-2)-(n-3)]!(n-3)!}) + \left[\frac{n!}{(n-n)!n!}\right] = \frac{1}{2}\left[\binom{n}{o} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-2} + \dots + \binom{n}{n}\right]\left(\frac{(n-2)(n-3)!}{(n-3)!}\right) + 1$

$= \frac{1}{2}. 2^n(n-2) + 1 = 2^{n-1}(n-2) + 1.$

Therefore, $Rank(S) = \left[\frac{1}{2}\sum_{p=0}^{n}\binom{n}{p}\right]\binom{n-2}{n-3} + \binom{n}{n} = 2^{n-1}(n-2) + 1.$ □

THEOREM 11

Let $IDPO_n = S$ be the identity difference partial order preserving transformation semigroup.

Rank(S) = $|\mathcal{R}| - 2^n = (n-2)2^{(n-1)} + 1$.

Proof

Let $\alpha_1 S, \dots \alpha_{2^n + (2^{n-1})(n^2-n)} S\ (where\ |S| = 2^n + (2^{n-1})(n^2-n))$, and $\alpha_1, \alpha_2, \dots, \alpha_{2^n+(2^{n-1})(n^2-n)}$ are arbitrary elements of S. Suppose there are $n. 2^{(n-1)} + 1\ set\ of\ \mathcal{R}$ in $S\ \forall\ n \geq 3$, then the sets must contain the partial identity maps say $\alpha_1 = \left(\begin{smallmatrix}\{X_1\ X_2\} & \dots & X_n \\ i & & -\end{smallmatrix}\right), \alpha_2 = \left(\begin{smallmatrix}\{X_1\ X_2\} & \dots & X_n \\ i+1 & & -\end{smallmatrix}\right), \dots \alpha_n = \left(\begin{smallmatrix}\{X_1\ X_n & \dots & X_n \\ n & & -\end{smallmatrix}\right)$, the identity maps say $\beta_1 = \left(\begin{smallmatrix}\{X_1 \dots X_n\} \\ i\end{smallmatrix}\right) \dots \beta_n \left(\begin{smallmatrix}\{X_1, \dots X_n\} \\ n\end{smallmatrix}\right)\ and\ the\ empty\ map\ \xi = \left(\begin{smallmatrix}\{X_1 \dots X_n\} \\ -\end{smallmatrix}\right)$ where $\{\alpha_1, \alpha_2, \dots \alpha_n\}, \{\beta_1, \beta_2, \dots, \beta_n\}, \ . \ . \ . \ , \{\xi\} \in \mathcal{R}$.

Let $|\emptyset|$ be the order of the set of $\mathcal{R}$ containing the (partial) identity and the empty map elements, then by combinatorial analysis, $|\emptyset| = 2^n$.

Since in S, the $|\mathcal{R}| = n. 2^{(n-1)} + 1$ and $|\emptyset| = 2^n$. Then

$Rank(S) = |\mathcal{R}| - |\emptyset| = \left(n.2^{(n-1)} + 1\right) - 2^n = n.2^n.2^{-1} + 1 - 2^n = n.2^n.2^{-1} - 2^n + 1 = 2^n(n.2^{-1} - 1) + 1 = 2^n\left(\frac{n}{2} - 1\right) + 1 = 2^n\left(\frac{n-2}{2}\right) + 1 \quad = 2^n.2^{-1}(n-2) + 1 = 2^{(n-1)}(n-2) + 1.$

See table below;

$n \geq 3$	$\lvert\mathcal{R}\rvert = n.2^{n-1} + 1$	$Rank(S) = 2^{(n-1)}(n-2) + 1$	$\lvert Rank - \lvert\mathcal{R}\rvert\rvert = 2^n$
3	13	5	8
4	33	17	16
5	81	49	32
6	193	129	64
7	449	321	128
8	1025	769	256
9	2305	1793	512
10	5121	4097	1024

V. CONCLUSION

We therefore conclude that the rank of identity difference transformation semigroup exist and can be easily obtained using the $R - classes$ of the respective subsemigroups as shown in section 3 and 4.

Conflict of Interest Statement:
This is to affirm that there is no conflict of interest amongst the authors or whosoever.

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