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On the convergence of two iterative methods for k-strictly pseudo-contractive mappings in CAT(0) spaces

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Abstract— In this paper, we prove the demiclosedness principle for k -strictly pseudo-contractive mappings and establish the Δ - convergence theorem of the cyclic algorithm for such mappings in CAT(0) spaces. Also, we give the strong convergence theorem of the modified Halpern iteration for k -strictly pseudo-contractive mappings in CAT(0) spaces. Our results extend and improve the corresponding recent results announced by many authors in the literature.

Keywords— CAT(0) space, fixed point, k -strictly pseudocontractive mapping, iterative method.

I. Introduction

LET C be a nonempty subset of a Hilbert space X. Recall that a mapping $T:C\to C$ is said to be k-strictly pseudo-contractive if there exists a constant $k\in[0,1)$ such that

$$\|Tx-Ty\|^2\le\|x-y\|^2+k\|(I-T)x-(I-T)y\|^2,\ \forall x,y\in C.$$

A point $x\in C$ is called a fixed point of T if $x=Tx$. We will denote the set of fixed points of T by $F(T)$. Note that the class of k-strictly pseudo-contractions includes the class of nonexpansive mappings T on C as a subclass. That is, T is nonexpansive if and only if T is 0-strictly pseudo-contractive. The mapping T is also said to be pseudo-contractive if $k=1$ and T is said to be strongly pseudo-contractive if there exists a constant $\lambda\in(0,1)$ such that $T-\lambda I$ is pseudo-contractive. Clearly, the class of k-strictly pseudo-contractive mappings is the one between classes of nonexpansive mappings and pseudo-contractive mappings. Also we remark that the class of strongly pseudo-contractive mappings is independent from the class of k-strictly pseudo-contractive mappings. Recently, many authors have been devoting the studies on the problems of finding fixed points for k-strictly pseudo-contractive mappings (see, e.g., [1]- [3]).

We define the concept of k-strictly pseudo-contractive mapping in a CAT(0) space as follows.

Let C be a nonempty subset of a CAT(0) space X. A mapping $T:C\to C$ is said to be k-strictly pseudo-contractive if there exists a constant $k\in[0,1)$ such that

$$d(Tx,Ty)^2\le d(x,y)^2+k\big(d(x,Tx)+d(y,Ty)\big)^2,\ \forall x,y\in C. \quad (1)$$

Acedo and Xu [4] introduced a cyclic algorithm in a Hilbert space. We modify this algorithm in a CAT(0) space.

Let $x_0\in C$ and $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a,b\in(0,1)$. The cyclic algorithm generates a sequence $\{x_n\}$ in the following way:

$$\begin{cases} x_1=\alpha_0x_0\oplus(1-\alpha_0)T_0x_0,\\ x_2=\alpha_1x_1\oplus(1-\alpha_1)T_1x_1,\\ \quad\vdots\\ x_N=\alpha_{N-1}x_{N-1}\oplus(1-\alpha_{N-1})T_{N-1}x_{N-1},\\ x_{N+1}=\alpha_Nx_N\oplus(1-\alpha_N)T_0x_N,\\ \quad\vdots\end{cases}$$

or, shortly,

$$x_{n+1}=\alpha_nx_n\oplus(1-\alpha_n)T_{[n]}x_n,\ \forall n\ge0, \quad (2)$$

where $T_{[n]}=T_i$, with $i=n\ (modN)$, $0\le i\le N-1$. By taking $T_{[n]}=T$ for all n in (2), we obtain the Mann iteration in [5].

In this paper, motivated by the above results, we prove the demiclosedness principle for k-strictly pseudo-contractive mappings in a CAT(0) space. Also we present the strong and Δ-convergence theorems of the cyclic algorithm and the modified

Halpern iteration which is introduced for Hilbert space by Hu [6] for such mappings in a CAT(0) space.

II. Preliminaries on CAT(0) space

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [7]), Euclidean buildings (see [8]), $\mathbb{R}$-trees (see [9]), the complex Hilbert ball with a hyperbolic metric (see [10]) and many others. For a throughout discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [7].

Fixed point theory in a CAT(0) space has been first studied by Kirk (see [11], [12]). He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory in a CAT(0) space has been rapidly developed and many papers have appeared (see e.g., [13]-[16]). It is worth mentioning that fixed point theorems in a CAT(0) space (specially in $\mathbb{R}$-trees) can be applied to graph theory, biology and computer science (see, e.g., [9], [17]-[20]).

Let (X,d) be a metric space. A *geodesic path* joining $x\in X$ to $y\in X$ (or more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l]\subset\mathbb{R}$ to X such that $c(0)=x$, $c(l)=y$ and $d(c(t),c(t'))=|t-t'|$ for all $t,t'\in[0,l]$. In particular, c is an isometry and $d(x,y)=l$. The image of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic and X is said to be a *uniquely geodesic* if there is exactly one geodesic joining x to y for each $x,y\in X$.

A *geodesic triangle* $\Delta(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consist of three points in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\Delta}(x_1,x_2,x_3)=\Delta(\overline{x}_1,\overline{x}_2,\overline{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x}_i,\overline{x}_j)=d(x_i,x_j)$ for $i,j\in\{1,2,3\}$. Such a triangle always exists (see [7]).

A geodesic metric space is said to be a CAT(0) space [7] if all geodesic triangles of appropriate size satisfy the following comparison axiom:

Let Δ be a geodesic triangle in X and $\overline{\Delta}$ be a comparison triangle for Δ. Then, Δ is said to satisfy the CAT(0) *inequality* if for all $x,y\in\Delta$ and all comparison points $\overline{x},\overline{y}\in\overline{\Delta}$,

$$d(x,y)\le d_{\mathbb{R}^2}(\overline{x},\overline{y}).$$

If x,y_1,y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1,y_2]$, then the CAT(0) inequality implies that

$$d(x,y_0)^2\le\frac{1}{2}d(x,y_1)^2+\frac{1}{2}d(x,y_2)^2-\frac{1}{4}d(y_1,y_2)^2.$$

This is the (CN) inequality of Bruhat and Tits [21]. In fact (see [7], p.163), a geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality. It is worth mentioning that the results in a CAT(0) space can be applied to any $CAT(k)$ space with $k\le 0$ since any $CAT(k)$ space is a $CAT(k')$ space for every $k'\ge k$ (see [7], p.165).

Let $x,y\in X$ and by Lemma 2.1 (iv) of [13] for each $t\in[0,1]$, there exists a unique point $z\in[x,y]$ such that

$$d(x,z)=td(x,y),\ \ d(y,z)=(1-t)d(x,y). \tag{3}$$

From now on, we will use the notation $(1-t)x\oplus ty$ for the unique point z satisfying (3). We now collect some elementary facts about CAT(0) spaces which will be used in sequel the proofs of our main results.

Lemma 1 *Let X be a CAT(0) space. Then*
(i) (see [13], Lemma 2.4) for each $x,y,z\in X$ and $t\in[0,1]$, one has

$$d((1-t)x\oplus ty,z)\le(1-t)d(x,z)+td(y,z),$$

(ii) (see [13], Lemma 2.5) for each $x,y,z\in X$ and $t\in[0,1]$, one has

$$d((1-t)x\oplus ty,z)^2\le(1-t)d(x,z)^2+td(y,z)^2-t(1-t)d(x,y)^2.$$

III. Demiclosedness principle for k-strictly pseudo-contractive mappings

In 1976 Lim [22] introduced a concept of convergence in a general metric space setting which is called Δ-convergence. Later, Kirk and Panyanak [23] used the concept of Δ-convergence introduced by Lim [22] to prove on the CAT(0) space analogs of some Banach space results which involve weak convergence. Also, Dhompongsa and Panyanak [13] obtained the Δ-convergence theorems for the Picard, Mann and Ishikawa iterations in a CAT(0) space for nonexpansive mappings under some appropriate conditions.

We now give the definition and collect some basic properties of the Δ-convergence.

Let X be a complete CAT(0) space and $\{x_n\}$ be a bounded sequence in X. For $x\in X$, we set

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$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\}.$$

The *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}.$$

It is known that in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point (see [24], Proposition 7).

Definition 1 *([22], [23]) A sequence $\{x_n\}$ in a CAT(0) space X is said to be Δ-convergent to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and x is called the Δ-limit of $\{x_n\}$.*

Lemma 2

i) Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence. (see [23], p.3690)

ii) Let C be a nonempty closed convex subset of a complete CAT(0) space and let $\{x_n\}$ be a bounded sequence in C. Then the asymptotic center of $\{x_n\}$ is in C. (see [25], Proposition 2.1)

Lemma 3 *([13], Lemma 2.8) If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n,u)\}$ is convergent then $x = u$.*

Let C be a closed convex subset of a CAT(0) space X and $\{x_n\}$ be a bounded sequence in C. We denote the notation

$$\{x_n\} \mapsto w \Leftrightarrow \Phi(w) = \inf_{x\in C} \Phi(x) \tag{4}$$

where $\Phi(x) = \limsup_{n\to\infty} d(x_n,x)$.

Nanjaras and Panyanak [26] gave a connection between the "$\mapsto$" convergence and Δ-convergence.

Proposition 1 *([26], Proposition 3.12) Let C be a closed convex subset of a CAT(0) space X and $\{x_n\}$ be a bounded sequence in C. Then $\Delta\text{-}\lim_{n\to\infty} x_n = p$ implies that $\{x_n\} \mapsto p$.*

The purpose of this section is to prove *demiclosedness principle for k-strictly pseudo-contractive mappings in a CAT(0) space* by using the convergence defined in (4).

Theorem 1 *Let C be a nonempty closed convex subset of a complete CAT(0) space X and $T: C \to C$ be a k-strictly pseudo-contractive mapping such that $k \in \left[0, \frac{1}{2}\right)$ and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a bounded sequence in C such that $\Delta\text{-}\lim_{n\to\infty} x_n = w$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $Tw = w$.*

Proof By the hypothesis, $\Delta\text{-}\lim_{n\to\infty} x_n = w$. From Proposition 1, we get $\{x_n\} \mapsto w$. Then we obtain $A(\{x_n\}) = \{w\}$ by Lemma 2 (ii) (see [26]). Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then we get

$$\Phi(x) = \limsup_{n\to\infty} d(x_n,x) = \limsup_{n\to\infty} d(Tx_n,x) \tag{5}$$

for all $x \in C$. In (5) by taking $x = Tw$, we have

$$\begin{aligned}
\Phi(Tw)^2 &= \limsup_{n\to\infty} d(Tx_n,Tw)^2 \\
&\leq \limsup_{n\to\infty}\left\{d(x_n,w)^2 + k(d(x_n,Tx_n) + d(w,Tw))^2\right\} \\
&\leq \limsup_{n\to\infty} d(x_n,w)^2 + k \limsup_{n\to\infty}\left(d(x_n,Tx_n) + d(w,Tw)\right)^2 \\
&= \Phi(w)^2 + kd(w,Tw)^2
\end{aligned} \tag{6}$$

The (CN) inequality implies that

$$d\left(x_n, \frac{w \oplus Tw}{2}\right)^2 \leq \frac{1}{2} d(x_n,w)^2 + \frac{1}{2} d(x_n,Tw)^2 - \frac{1}{4} d(w,Tw)^2.$$

Letting $n \to \infty$ and taking superior limit on the both sides of the above inequality, we get

$$\Phi\left(\frac{w \oplus Tw}{2}\right)^2 \leq \frac{1}{2}\Phi(w)^2 + \frac{1}{2}\Phi(Tw)^2 - \frac{1}{4} d(w,Tw)^2.$$

Since $A(\{x_n\}) = \{w\}$, we have

$$\Phi(w)^2 \leq \Phi\left(\frac{w \oplus Tw}{2}\right)^2 \leq \frac{1}{2}\Phi(w)^2 + \frac{1}{2}\Phi(Tw)^2 - \frac{1}{4} d(w,Tw)^2.$$

which implies that

$$d(w,Tw)^2 \leq 2\Phi(Tw)^2 - 2\Phi(w)^2. \tag{7}$$

By (6) and (7), we get $(1-2k)d(w,Tw)^2 \leq 0$. Since $k \in \left[0, \frac{1}{2}\right)$, then we have $Tw = w$ as desired.

Now, we prove the Δ-convergence of the cyclic algorithm for k-strictly pseudo-contractive mappings in a CAT(0) space.

Theorem 2 *Let C be a nonempty closed convex subset of a complete* CAT(0) *space X and $N \geq 1$ be an integer. Let, for each $0 \leq i \leq N-1$, $T_i : C \to C$ be k_i-strictly pseudo-contractive mappings for some $0 \leq k_i < \frac{1}{2}$. Let $k = \max\{k_i; 0 \leq i \leq N-1\}$, $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$ and $k < a$. Let $F = \cap_{i=0}^{N-1} F(T_i) \neq \varnothing$. For $x_0 \in C$, let $\{x_n\}$ be a sequence defined by (2). Then the sequence $\{x_n\}$ is Δ-convergent to a common fixed point of the family $\{T_i\}_{i=0}^{N-1}$.*

Proof Let $p \in F$. Using (1), (2) and Lemma 1, we have

$$\begin{aligned} d(x_{n+1}, p)^2 &= d(\alpha_n x_n \oplus (1-\alpha_n) T_{[n]} x_n, p)^2 \\ &\leq \alpha_n d(x_n, p)^2 + (1-\alpha_n) d(T_{[n]} x_n, p)^2 \\ &\quad - \alpha_n (1-\alpha_n) d(x_n, T_{[n]} x_n)^2 \\ &\leq \alpha_n d(x_n, p)^2 + (1-\alpha_n)\left\{d(x_n, p)^2 + k d(x_n, T_{[n]} x_n)^2\right\} \\ &\quad - \alpha_n (1-\alpha_n) d(x_n, T_{[n]} x_n)^2 \\ &= d(x_n, p)^2 - (1-\alpha_n)(\alpha_n - k) d(x_n, T_{[n]} x_n)^2 \qquad (8) \\ &\leq d(x_n, p)^2. \end{aligned}$$

This inequality guarentees that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$. By (8), we also have

$$\begin{aligned} d(x_n, T_{[n]} x_n)^2 &\leq \frac{1}{(1-\alpha_n)(\alpha_n - k)} \left[d(x_n, p)^2 - d(x_{n+1}, p)^2\right] \\ &\leq \frac{1}{(1-b)(a-k)} \left[d(x_n, p)^2 - d(x_{n+1}, p)^2\right] \end{aligned}$$

Since $\lim_{n\to\infty} d(x_n, p)$ exists, we obtain $\lim_{n\to\infty} d(x_n, T_{[n]} x_n) = 0$. To show that the sequence $\{x_n\}$ is Δ-convergent to a common fixed point of the family $\{T_i\}_{i=0}^{N-1}$, we prove that

$$\omega_w(x_n) = \bigcup_{\{u_n\} \subseteq \{x_n\}} A(\{u_n\}) \subseteq F$$

and $\omega_w(x_n)$ consists of exactly one point. Let $u \in \omega_w(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim_{n\to\infty} v_n = v \in C$. By Theorem 1, we have $v \in F$ and by Lemma 3, we have $u = v \in F$. This shows that $\omega_w(x_n) \subseteq F$. Now we prove that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in F$. Finally, since $\{d(x_n, v)\}$ is convergent, we have $x = v \in F$ by Lemma 3. This completes the proof.

IV. The Strong Convergence Theorem for the Modified Halpern Iteration

In [6], Hu introduced a modified Halpern iteration. We modify this iteration in CAT(0) spaces as follows.

For an arbitary initial value $x_0 \in C$ and a fixed anchor $u \in C$, the sequence $\{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = \alpha_n u \oplus (1-\alpha_n) y_n, \\ y_n = \dfrac{\beta_n}{1-\alpha_n} x_n \oplus \dfrac{\gamma_n}{1-\alpha_n} T x_n, \ \forall n \geq 0, \end{cases} \qquad (9)$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three real sequences in $(0,1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$. Clearly, the iterative sequence (9) is a natural generalization of the well known iterations.

(i) If we take $\beta_n = 0$ for all n in (9), then the sequence (9) reduces to the Halpern's iteration in [27].

(ii) If we take $\alpha_n = 0$ for all n in (9), then the sequence (9) reduces to the Mann iteration in [5].

In this section, we prove the strong convergence of the modified Halpern's iteration in a CAT(0) space.

Recall that a continous linear functional μ on ℓ_∞, the Banach space of bounded real sequences, is called a Banach limit if $\|\mu\| = \mu(1,1,...) = 1$ and $\mu(a_n) = \mu(a_{n+1})$ for all $\{a_n\}_{n=1}^\infty \subset \ell_\infty$.

Lemma 4 *(see [28], Proposition 2) Let $\{a_1, a_2, ...\} \in \ell_\infty$ be such that $\mu(a_n) \leq 0$ for all Banach limits μ and $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$. Then, $\limsup_{n\to\infty} a_n \leq 0$.*

Lemma 5 *Let C be a nonempty closed convex subset of a complete* CAT(0) *space X, $T: C \to C$ be a k-strictly pseudo-contractive mapping with $k \in [0,1)$ and $S: C \to C$ be a mapping defined by $Sz = kz \oplus (1-k)Tz$, for $z \in C$. Let $u \in C$ be fixed. For each $t \in [0,1]$, the mapping $S_t : C \to C$ defined by*

$$S_t z = tu \oplus (1-t) Sz = tu \oplus (1-t)(kz \oplus (1-k)Tz), \text{ for } z \in C,$$

has a unique fixed point $z_t \in C$, that is,

$$z_t = S_t(z_t) = tu \oplus (1-t) S(z_t). \qquad (10)$$

Proof As it has been proven in [29], if T is a k-strictly pseudo-contractive mapping with $k \in [0,1)$, S is a nonexpansive mapping such that $F(S) = F(T)$. Then, from

Lemma 2.1 in [14], the mapping S_t has a unique fixed point $z_t \in C$.

Lemma 6 *Let X, C, T and S be as in Lemma 5. Then, $F(T) \neq \emptyset$ if and only if $\{z_t\}$ given by (10) remains bounded as $t \to 0$. In this case, the following statements hold:*
1) $\{z_t\}$ converges to the unique fixed point z of T which is nearest to u,
2) $d^2(u,z) \leq \mu\ d^2(u,x_n)$ for all Banach limits μ and all bounded sequences $\{x_n\}$ with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof If $F(T) \neq \emptyset$, then we have $F(S) = F(T) \neq \emptyset$. Also, if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we obtain that

$$\begin{aligned} d(x_n, Sx_n) &= d(x_n, kx_n \oplus (1-k)Tx_n) \\ &\leq (1-k)d(x_n, Tx_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, from Lemma 2.2 in [14], the rest of the proof of this lemma can be seen.

The following lemma can be found in [30].

Lemma 7 *(see [30], Lemma 2.1) Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the condition*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n \sigma_n, \ \forall n \geq 0,$$

where $\{\gamma_n\}$ and $\{\sigma_n\}$ are sequences of real numbers such that
(1) $\{\gamma_n\} \subset [0,1]$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$,
(2) either $\limsup_{n\to\infty} \sigma_n \leq 0$ or $\sum_{n=1}^{\infty} |\gamma_n \sigma_n| < \infty$.
Then, $\lim_{n\to\infty} a_n = 0$.

We are now ready to prove our main result.

Theorem 3 *Let C be a nonempty closed convex subset of a complete* CAT(0) *space X and $T : C \to C$ be a k-strictly pseudo-contractive mapping such that $0 \leq k < \frac{\beta_n}{1-\alpha_n} < 1$ and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (9). Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:*

C1) $\lim_{n\to\infty} \alpha_n = 0$,
C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
C3) $\lim_{n\to\infty} \beta_n \neq k$ and $\lim_{n\to\infty} \gamma_n \neq 0$..

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof We divide the proof into three steps. In the first step we show that $\{x_n\}, \{y_n\}$ and $\{Tx_n\}$ are bounded sequences. In the second step we show that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Finally, we show that $\{x_n\}$ converges to a fixed point $z \in F(T)$ which is nearest to u.
First step: Take any $p \in F(T)$, then, from Lemma 1 and (9), we have

$$\begin{aligned} & d(y_n, p)^2 \\ & \leq \frac{\beta_n}{1-\alpha_n} d(x_n, p)^2 + \frac{\gamma_n}{1-\alpha_n} d(Tx_n, p)^2 - \frac{\beta_n \gamma_n}{(1-\alpha_n)^2} d(x_n, Tx_n)^2 \\ & \leq \frac{\beta_n}{1-\alpha_n} d(x_n, p)^2 + \frac{\gamma_n}{1-\alpha_n} \left(d(x_n, p)^2 + kd(x_n, Tx_n)^2 \right) \\ & \quad - \frac{\beta_n \gamma_n}{(1-\alpha_n)^2} d(x_n, Tx_n)^2 \\ & = d(x_n, p)^2 - \frac{\gamma_n}{1-\alpha_n} \left(\frac{\beta_n}{1-\alpha_n} - k \right) d(x_n, Tx_n)^2 \\ & \leq d(x_n, p)^2. \end{aligned}$$

Also, we obtain

$$\begin{aligned} & d(x_{n+1}, p)^2 \\ & \leq \alpha_n d(u, p)^2 + (1-\alpha_n) d(y_n, p)^2 - \alpha_n (1-\alpha_n) d(u, y_n)^2 \\ & \leq \alpha_n d(u, p)^2 \\ & \quad + (1-\alpha_n) \left\{ d(x_n, p)^2 - \frac{\gamma_n}{1-\alpha_n} \left(\frac{\beta_n}{1-\alpha_n} - k \right) d(x_n, Tx_n)^2 \right\} \\ & \quad - \alpha_n (1-\alpha_n) d(u, y_n)^2 \\ & = \alpha_n d(u, p)^2 + (1-\alpha_n) d(x_n, p)^2 - \gamma_n \left(\frac{\beta_n}{1-\alpha_n} - k \right) d(x_n, Tx_n)^2 \\ & \quad - \alpha_n (1-\alpha_n) d(u, y_n)^2 \qquad (11) \\ & \leq \alpha_n d(u, p)^2 + (1-\alpha_n) d(x_n, p)^2 \\ & \leq \max\left\{ d(u, p)^2, d(x_n, p)^2 \right\}. \end{aligned}$$

By induction,

$$d(x_{n+1}, p)^2 \leq \max\left\{ d(u, p)^2, d(x_0, p)^2 \right\}.$$

This proves the boundedness of the sequence $\{x_n\}$, which leads to the boundedness of $\{Tx_n\}$ and $\{y_n\}$.
Second step: In fact, we have from (11) (for some appropriate constant $M > 0$) that

$$
\begin{aligned}
&d(x_{n+1}, p)^2 \\
&\le \alpha_n d(u,p)^2 + (1-\alpha_n) d(x_n, p)^2 - \gamma_n\left(\frac{\beta_n}{1-\alpha_n} - k\right) d(x_n, Tx_n)^2 \\
&= \alpha_n (d(u,p)^2 - d(x_n,p)^2) + d(x_n,p)^2 \\
&\quad - \gamma_n\left(\frac{\beta_n}{1-\alpha_n} - k\right) d(x_n, Tx_n)^2 \\
&\le \alpha_n M + d(x_n,p)^2 - \gamma_n\left(\frac{\beta_n}{1-\alpha_n} - k\right) d(x_n, Tx_n)^2,
\end{aligned}
$$

which implies that

$$\gamma_n\left(\frac{\beta_n}{1-\alpha_n} - k\right) d(x_n, Tx_n)^2 - \alpha_n M \le d(x_n,p)^2 - d(x_{n+1},p)^2. \tag{12}$$

If $\gamma_n\left(\frac{\beta_n}{1-\alpha_n} - k\right) d(x_n, Tx_n)^2 - \alpha_n M \le 0$, then

$$d(x_n, Tx_n)^2 \le \frac{\alpha_n}{\gamma_n\left(\frac{\beta_n}{1-\alpha_n} - k\right)} M,$$

and hence the desired result is obtained by the conditions (C1) and (C3).

If $\gamma_n\left(\frac{\beta_n}{1-\alpha_n} - k\right) d(x_n, Tx_n)^2 - \alpha_n M > 0$, then following (12), we have

$$
\begin{aligned}
&\sum_{n=0}^{m}\left[\gamma_n\left(\frac{\beta_n}{1-\alpha_n} - k\right) d(x_n, Tx_n)^2 - \alpha_n M\right] \\
&\le d(x_0,p)^2 - d(x_{m+1},p)^2 \\
&\le d(x_0,p)^2.
\end{aligned}
$$

That is

$$\sum_{n=0}^{\infty}\left[\gamma_n\left(\frac{\beta_n}{1-\alpha_n} - k\right) d(x_n, Tx_n)^2 - \alpha_n M\right] < \infty.$$

Thus

$$\lim_{n\to\infty}\left[\gamma_n\left(\frac{\beta_n}{1-\alpha_n} - k\right) d(x_n, Tx_n)^2 - \alpha_n M\right] = 0.$$

Then we get

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{13}$$

Third step: Using the condition (C1) and (13), we obtain

$$
\begin{aligned}
&d(x_{n+1}, x_n) \le d(x_{n+1}, Tx_n) + d(Tx_n, x_n) \\
&\le \alpha_n d(u, Tx_n) + (1-\alpha_n) d(y_n, Tx_n) + d(Tx_n, x_n) \\
&\le \alpha_n d(u, Tx_n) + (1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n} d(x_n, Tx_n)\right) + d(Tx_n, x_n) \\
&= \alpha_n d(u, Tx_n) + (\beta_n + 1) d(x_n, Tx_n) \\
&\to 0, \text{ as } n\to\infty.
\end{aligned}
$$

Also, from (13), we have

$$d(x_n, y_n) \le \frac{\gamma_n}{1-\alpha_n} d(x_n, Tx_n) \to 0, \text{ as } n\to\infty. \tag{14}$$

Let $z = \lim_{t\to 0} z_t$, where z_t is given by (10) in Lemma 5. Then, z is the point of $F(T)$ which is nearest to u. By Lemma 6 (2), we have $\mu(d(u,z)^2 - d(u,x_n)^2) \le 0$ for all Banach limits μ. Moreover, since $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$,

$$\limsup_{n\to\infty}\left[\left(d(u,z)^2 - d(u,x_{n+1})^2\right) - \left(d(u,z)^2 - d(u,x_n)^2\right)\right] = 0.$$

If we take $a_n = d(u,z)^2 - d(u,x_n)^2$ in Lemma 4, then we obtain

$$\limsup_{n\to\infty}\left(d(u,z)^2 - d(u,x_n)^2\right) \le 0. \tag{15}$$

It follows from the condition (C1) and (14) that

$$
\begin{aligned}
&\limsup_{n\to\infty}\left(d(u,z)^2 - (1-\alpha_n) d(u,y_n)^2\right) \\
&= \limsup_{n\to\infty}\left(d(u,z)^2 - d(u,x_n)^2\right).
\end{aligned} \tag{16}
$$

By (15) and (16), we have

$$\limsup_{n\to\infty}\left(d(u,z)^2 - (1-\alpha_n) d(u,y_n)^2\right) \le 0. \tag{17}$$

We observe that

$$
\begin{aligned}
&d(x_{n+1}, z)^2 \\
&\le \alpha_n d(u,z)^2 + (1-\alpha_n) d(y_n,z)^2 - \alpha_n(1-\alpha_n) d(u,y_n)^2 \\
&\le \alpha_n d(u,z)^2 + (1-\alpha_n) d(x_n,z)^2 - \alpha_n(1-\alpha_n) d(u,y_n)^2 \\
&= (1-\alpha_n) d(x_n,z)^2 + \alpha_n[d(u,z)^2 - (1-\alpha_n) d(u,y_n)^2].
\end{aligned}
$$

It follows from the condition (C2) and (17), using Lemma 7, that $\lim_{n\to\infty} d(x_n, z) = 0$. This completes the proof of Theorem 3.

We obtain the following corollary as a direct consequence of Theorem 3.

Corollary 1 *Let* X, C *and* T *be as Theorem 3. Let* $\{\alpha_n\}$ *be a real sequence in* $(0,1)$ *satisfying the conditions (C1) and (C2). For a constant* $\delta \in (k,1)$, *an arbitary initial value* $x_0 \in C$ *and a fixed anchor* $u \in C$, *let the sequence* $\{x_n\}$ *be defined by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)\left(\delta x_n \oplus (1-\delta)Tx_n\right), \ \forall n \geq 0. \qquad (18)$$

Then the sequence $\{x_n\}$ *is strongly convergent to a fixed point of* T.

Proof If, in proof of Theorem 3, we take $\beta_n = (1-\alpha_n)\delta$ and $\gamma_n = (1-\alpha_n)(1-\delta)$, then we get the desired conclusion.

Remark 1 *The results in this section contain the strong convergence theorems of the iterative sequences (9) and (18) for nonexpansive mappings in a* CAT(0) *space. Also, these results contain the corresponding theorems proved for these iterative sequences in a Hilbert space.*

References

[1] I. Inchan, "Strong convergence theorems for a new iterative method of k-strictly pseudo-contractive mappings in Hilbert spaces," Computers and Mathematics with Applications, vol. 58, pp. 1397-1407, 2009.

[2] J. S. Jung, "Some results on a general iterative method for k-strictly pseudo-contractive mappings," Fixed Point Theory and Applications, 2011:24, doi:10.1186/1687-1812-2011-24, 11 pages, 2011.

[3] S. Li, L. Li, L. Zhang and X. He, "Strong convergence of modified Halpern's iteration for a k-strictly pseudocontractive mapping," Journal of Inequalities and Applications, 2013:98, doi:10.1186/1029-242X-2013-98, 2013.

[4] G. L. Acedo and H. K. Xu, "Iterative methods for strict pseudo-contractions in Hilbert spaces," Nonlinear Anal., vol. 67, no. 7, pp. 2258-2271, 2007.

[5] W. R. Mann, "Mean value methods in iteration," Proceedings of the American Mathematical Society, vol. 4, pp. 506-513, 1953.

[6] L. G. Hu, "Strong convergence of a modified Halpern's iteration for nonexpansive mappings," Fixed Point Theory and Applications, vol. 2008, Article ID 649162, 9 pages, doi: 10.1155/2008/649162, 2008.

[7] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, vol. 319, Grundlehren der Mathematischen Wissenschaften, Springer, Berlin, Germany, 1999.

[8] K.S. Brown, *Buildings*, Springer, NewYork, NY, USA, 1989.

[9] W. A. Kirk, "Fixed point theorems in CAT(0) spaces and $\mathbb{R}$-trees," Fixed Point Theory and Applications, vol. 2004, no. 4, pp. 309–316, 2004.

[10] K. Goebel and S. Reich, *Uniform Convexity*, *Hyperbolic Geometry and Nonexpansive Mappings*, vol. 83 of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker Inc., New York, NY, USA, 1984.

[11] W. A. Kirk, "Geodesic geometry and fixed point theory," in Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), vol. 64 of Coleccion Abierta, pp. 195–225, University of Seville, Secretary of Publications, Seville, Spain, 2003.

[12] W. A. Kirk," Geodesic geometry and fixed point theory II," in International Conference on Fixed Point Theory and Applications, pp. 113–142, Yokohama Publishers, Yokohama, Japan, 2004.

[13] S. Dhompongsa and B. Panyanak, "On Δ-convergence theorems in CAT(0) spaces," Computers and Mathematics with Applications, vol. 56, no. 10, pp. 2572–2579, 2008.

[14] S. Saejung, "Halpern's iteration in CAT(0) spaces," Fixed Point Theory and Applications, vol. 2010, Article ID 471781, 13 pages, 2010.

[15] A. Şahin and M. Başarır, "On the strong and Δ-convergence theorems for nonself mappings on a CAT(0) space," Proceedings of the 10th IC-FPTA, July 9-18 2012, Cluj-Napoca, Romania, pp. 227-240, 2012.

[16] A. Şahin and M. Başarır, "On the strong convergence of a modified S-iteration process for asymptotically quasi-nonexpansive mappings in a CAT(0) space," Fixed Point Theory and Applications, 2013:12, doi:10.1186/1687-1812-2013-12, 10 pages, 2013.

[17] M. Bestvina, "R-trees in topology, geometry, and group theory," in Handbook of Geometric Topology, pp. 55–91, North-Holland, Amsterdam, The Netherlands, 2002.

[18] R. Espinola and W. A. Kirk, "Fixed point theorems in R-trees with applications to graph theory," Topology and Its Applications, vol. 153, no. 7, pp. 1046–1055, 2006.

[19] W. A. Kirk, "Some recent results in metric fixed point theory," Journal of Fixed Point Theory and Applications, vol. 2, no. 2, pp. 195–207, 2007.

[20] C. Semple and M. Steel, *Phylogenetics*, vol. 24 of Oxford Lecture Series in Mathematics and Its Applications, Oxford University Press, Oxford, UK, 2003.

[21] F. Bruhat and J. Tits, "Groupes réductifs sur un corps local," Inst. Hautes Études Sci. Publ. Math., vol. 41, pp. 5–251, 1972.

[22] T. C. Lim, "Remarks on some fixed point theorems," Proceedings of the American Mathematical Society, vol. 60, pp. 179-182, 1976.

[23] W.A. Kirk and B. Panyanak, "A concept of convergence in geodesic spaces," Nonlinear Analysis: Theory, Methods and Applications, vol. 68, no. 12, pp. 3689–3696, 2008.

[24] S. Dhompongsa, W. A. Kirk and B. Sims, "Fixed point of uniformly lipschitzian mappings," Nonlinear Analysis: Theory, Methods and Applications, vol. 65, no.4, pp. 762–772, 2006.

[25] S. Dhompongsa, W. A. Kirk and B. Panyanak, "Nonexpansive set-valued mappings in metric and Banach spaces," Journal of Nonlinear and Convex Analysis, vol. 8, no. 1, pp. 35–45, 2007.

[26] B. Nanjaras and B. Panyanak, "Demiclosed principle for asymptotically nonexpansive mappings in CAT(0) spaces," Fixed Point Theory and Applications, vol. 2010, Article ID 268780, doi: 10.1155/2010/268780, 14 pages, 2010.

[27] B. Halpern, "Fixed points of nonexpanding maps," Bulletin of the American Mathematical Society, vol. 73, no. 6, pp. 957–961, 1967.

[28] N. Shioji and W. Takahashi, "Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces," Proceedings of the American Mathematical Society, vol. 125, no. 12, pp. 3641–3645, 1997.

[29] H. Zhou, "Convergence theorems of fixed points for k-strict pseudo-contractions in Hilbert space," Nonlinear Anal., vol. 69, pp. 456-462, 2008.

[30] H. K. Xu, "An iterative approach to quadratic optimization," Journal of Optimization Theory and Applications, vol. 116, no. 3, pp. 659–678, 2003.