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On General Product of Two Finite Cyclic Groups one being of order 7 (Induced by = (1) (2) (3) (4) (5) (6) (7))

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Abstract—In this paper we find the general product induced by the semi special permutation π = (1)(2)(3)(4)(5)(6)(7). That is the general products of two finite cyclic groups in which one of order 7 and the other is of order m these general products can be described in terms of numerical parameters.

Keywords— semi special permutations, general product

I. INTRODUCTION

If A, B are two subgroups of a group G then we say that G is the general product of A, B if and only if:
(1) G = AB
(2) A, B has no elements in common other than the identity i.e. $A\bigcap B=\{e\}$.

Now if A = {a} is a cyclic group of order m, B = {b} is a cyclic group of order n then there exist corresponding to G two semi special permutations π, ρ where π on [n], ρ on [m] such that

$$a^y b^x = b^{\pi^y x} a^{\rho^x y}, x\in[n],\ y\in[m] \quad \text{….. (1)}$$

$$\pi^m x \equiv x(mod\ \ n), x\in[n] \quad \text{….. (2)}$$

$$\rho^n y \equiv y(\bmod m), y\in[m] \quad \text{….. (3)}$$

Where *[c]* demote to the set of dements *{1, 2, 3, …, c}*

Definition: (Semi special permutation) *A* permutation π on *[c]* is said to be semi special on *[c] iff* π *(c) = c*, $\pi_z(x)=\pi(x+z)-\pi z$ (mod *c*), y∈[*c*] is a power depending on *z* of π

Theorem A:

(i) $a^m b^x = b^x a^m, x\in[n]$ …….. (4)

(ii) $a^y b^n = b^n a^y, y\in[m]$ …….. (5)

Theorem B:

(i) The order of π divides *m* i.e. if e is the orders of π then m is a multiple of ℓ.

(ii) There exist a number $\lambda,(\lambda,\frac{m}{\ell})=1$ thus that

$$a^\ell b = b a^{\ell\lambda}, \ell\lambda^\mu \equiv \ell \bmod m \quad \text{….. (6)}$$

Where μ is the **g.c.d** of all $\nu-\upsilon$; ν,υ are any numbers of the principal cycle of π .

(iii) $a^\ell b^\mu = b^\mu a^\ell$.

We know that $\pi=$ (1)(2)(3)(4)(5)(6)(7) is a semi special permutation on [7]

§1- The general product induced by π

Theorem 1.1: the defining relation of the general product of *G* corresponding to

$\pi=(1)(2)(3)(4)(5)(6)(7)$ is

$$G=\{a,b;a^m=b^7=e,ab=ba^r\} \quad \text{…….. (7.1)}$$

$$r^7\equiv 1(\bmod m) \quad \text{…….. (7.2)}$$

The converse is also true i.e. for any *r* satisfying (7.2) then any group *G* generated by *a* and *b* satisfying (7.1) is the general product of {*a*}, {*b*}.

Proof:

Assume that the general product of G exist, from the equation

$a^y b^y = b^{\pi^y x} a^{\rho^x y}, x\in[7], y\in[m]$, with *y*=1 we get

$ab^x = b^{\pi^1 x} a^{\rho^x 1}, x=1,2,3,4,5,6,7$ put *x*=1 then we have

$ab=b^{\pi 1}a^{\rho 1}$

let us write $\rho 1 = r$ then $ab=ba^r$,

$$ab^2 = abb = ba^r b = b\underbrace{aaa\ldots a}_{r\text{-times}}b$$

$ab^2=b^2a^{r^2}$ and so by induction we get

$$ab^7 = b^7 a^{r^7} \quad \text{….. (8)}$$

From theorem A with *n* = 7, *y* = 1

we have $ab^7=b^7a$ …..(9)

From 8, (9) we get $r^7\equiv 1(\bmod m)$ and so (7.2) follows.

Also we notice that {*a*} is of order m and {*b*} is of order 5 then 7.1 is the required defining relation of *G*.

The converse is also true to do this let G be a group generated by a, b with the defining relation (7.1) and satisfying the condition (7.2) and let $x = \{0, 1, 2, 3, 4, 5,6\}$, $y = \{0, 1, 2, \ldots, m-1\}$ and let H be the set of all ordered pairs (x, y) with $x \in X, y \in Y$ with binary operation $*$ defined on H as follows:

$(x, y) * (x', y') = (x'', y'')$ such that

$x'' = x + x' \pmod 7$

$y'' = r^{x'} y + y' \pmod m$

Then it is clear that $<H, *>$ is a group with $e = (0,0)$ as its identity element. Also if $\alpha = (0,1), \beta = (1,0)$ $\beta^x \alpha^y = (x, y)$ which implies that each element of H can be determined uniquely in the form $\beta^x \alpha^y$ which means that H is the general product of $\{\alpha\}, \{\beta\}$. since $\{\alpha\}$ is of order m and $\{\beta\}$ is of order 5 so $|H| = 7m$, it is evident to see that $\alpha^m = \beta^m = e$,$\alpha\beta = \beta\alpha^r$ which are corresponding to the defining relation of G and so the permutation $\pi = (1)(2)(3)(4)(5)(6)(7)$ is induced by α .

Also H can be considered as a homomorphic image of G, since $|G| \leq 7m$ and hence the two groups are isomorphic hence the theorem is proved.

Remark: It must be noted that two groups G, L with defining relation:

$$G = \{a, b; a^m = b^7 = e, ab = ba^r, r^7 \equiv 1 (\operatorname{mod} m)\}$$

$$L = \{a, b; a^m = b^7 = e, ab = ba^s, s^7 \equiv 1 (\operatorname{mod} m)\}$$

Such that $r \not\equiv s$ mod m, then $G \cong L$ if and only if $r \equiv s^6 (\operatorname{mod} m)$

Conclusion:

The general product of two finite cyclic groups one being of order 7, which is corresponding to $\pi = (1)(2)(3)(4)(5)(6)(7)$ is obtained by theorem 1.1 with defining relation (7.1), (7.2).

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