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Elliptic Curves Over The Quadratic Field

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Abstract—In this paper, we introduce some fundamental results of the elliptic curve over the quadratic field. After we create an elliptic curve over the quadratic field with an element of infinite order.[2,3,4].

Keywords—Elliptic Curves, Quadratic Fields, Infinite Order.

I. Introduction

We introduce some importants results over the ring of integers of the quadratic fields.

Definition 1. The quadratic field is any extension of degree two over the rational field $\mathbb{Q}$.

Theorem 2. All quadratic field is of the form $\mathbb{Q}(\sqrt{d})$, where d is an integer without square factor.

Proposition 3. Let $K=\mathbb{Q}(\sqrt{d})$, is a quadratic field where d is an integer without square factor.
1. If $d\equiv 2 \mod 4$ or $d\equiv 3 \mod 4$ then the integer ring of K is the set of $a+b\sqrt{d}$ where $a,b\in\mathbb{Z}$.
2. If $d\equiv 1 \mod 4$ then the integer ring of K is the set of $\frac{1}{2}(a+b\sqrt{d})$ where $a,b\in\mathbb{Z}$ and $a\equiv b \mod 2$.

Definition 4. An elliptic curve over the quadratic field $K=\mathbb{Q}(\sqrt{d})$, is curve that is given by Weierstrass equation: $Y^2Z=X^3+AXZ^2+BZ^3$, where $A,B\in K$. [1,5].

II. Elliptic Curves over the quadratic field whit an element of infinite order

PleaseIn this section we introduce some lemmas for created an elliptic Curves over quadratic field whit an element of infinite order. $E_{A,B}$ K

Let an elliptic curve over the quadratic field given by Weierstrass equation:

$$Y^2Z=X^3+AXZ^2+BZ^3, \text{ where } A,B\in K.$$

Lemma 1. Let $A,B\in\mathbb{Z}[i]$, $K=\mathbb{Q}[i]$ and $P=(x,y)$ an element of finite order in $E_{A,B}$.

If $(x,y)\in(\mathbb{Z}[i])^2$ then $y=0$ or $y^2\mid 4A^3+27B^2$.

Proof $E_{A,B}$

Let an elliptic curve over the quadratic field $K=\mathbb{Q}[i]$ given by Weierstrass equation: $y^2=x^3+Ax+B$,

with $A,B\in\mathbb{Z}[i]$.

Let $P=(x,y)\in E_{A,B}$. Suppose that P has finite order.

If $x,y\in\mathbb{Z}[i]$, then by Lutz Nagelle Theorem [2], we have:

if $y\neq 0$ then $y^2\mid 4A^3+27B^2$..

Lemma 2. Let an elliptic curve over the quadratic field $K=\mathbb{Q}[i]$ given by Weierstrass equation: $y^2=x^3+Ax+B$,

with $A,B\in\mathbb{Z}[i]$. $E_{A,B}$

Then, there exists $A',B'\in\mathbb{Z}[i]$ such that $|A'|\geq|A|$ and $|B'|\geq|B|$ which the elliptic curve have a point of infinite order.

Proof $E_{A,B}$

Let an elliptic curve over the quadratic field $K=\mathbb{Q}[i]$ given by Weierstrass equation: $y^2=x^3+Ax+B$,

with $A,B\in\mathbb{Z}[i]$.

We pose:

$$A'=-(3|A|+1)^2,$$
$$B'=(3|B|+3)^2,$$
$$x_1=3|A|+1,$$

and $y_1 = 3|B|+3.$

We have:

$$\begin{aligned} x_1^3 + A' x_1 + B' &= (3|A|+1)^3 - (3|A|+1)^2 \times (3|A|+1) + (3|B|+3)^2 \\ &= (3|A|+1)^3 - (3|A|+1)^3 + (3|B|+3)^2 \\ &= (3|B|+3)^2 \\ &= y_1^2 \end{aligned}$$

It's clair that $Q = (x_1, y_1) \in E_{A',B'}$.

Suppose that Q has finite order, so by lemma2.1 we have:

$$\begin{aligned} y_1^2 \mid 4A'^3 + 27B'^2 &\Rightarrow 3 \mid 4A'^3 + 27B'^2 \\ &\Rightarrow 3 \mid 4A'^3 \\ &\Rightarrow 3 \mid A' \end{aligned}$$

Which is absurd because: $A' = -(3|A|+1)^2$

Lemma 3. Let $K = \mathbb{Q}(\sqrt{d})$ and $E_{A,B}$ an elliptic curve over K given by Weierstrass equation:

$$y^2 = x^3 + Ax + B, \quad \text{with } A, B \in \mathbb{Z}[\sqrt{d}].$$

Then, there exists $A', B' \in \mathbb{Z}[\sqrt{d}]$ such that $|A'| \geq |A|$ and $|B'| \geq |B|$ which the elliptic curve $E_{A',B'}$ over K have a point of an infinite order.

Proof Let $K = \mathbb{Q}(\sqrt{d})$ and $E_{A,B}$ an elliptic curve over K given by Weierstrass equation:

$$y^2 = x^3 + Ax + B, \quad \text{with } A, B \in \mathbb{Z}[\sqrt{d}].$$

We suppose:

$$T = sup\{|A|+1; |B|+1\},$$

$$A' = 2T,$$

$$B' = 3^2 T^2,$$

$$x_1 = \frac{1}{3},$$

and $$y_1 = \frac{1+3^4 T}{3^3}.$$

We have:

$$\begin{aligned} x_1^3 + A' x_1 + B' &= \tfrac{1}{3^6} + \tfrac{2T}{3^2} + 3^2 T^2 \\ &= \tfrac{1+2\times 3^4 \times T + 3^8 T^2}{3^6} \\ &= (\tfrac{1++3^4 T}{3^3})^2 \\ &= y_1^2. \end{aligned}$$

Such that , $Q = (x_1, y_1) \in E_{A',B'}$, so by lemma2.1 we have: Q has an infinite order.

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