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On the geometry of luxury

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Abstract—A class of transcendental preferences is employed as an explicit representation of the luxury-necessity dichotomy which admits a smooth Cobb-Douglas limit. The analytical tractability of the model enables us to represent explicitly Marshallian demand and income elasticity of demand. The noncommutativity of scale and substitution effects, and measured by Lie brackets of the corresponding vector fields, is employed in order to define a measure of deviation from scale symmetry which is profoundly connected with Shephard's distance.

Keywords— Cobb-Douglas functions, homotheticity, luxury, scale effect, income effect, substitution effect, Lie bracket.

I. Introduction

IN 1928 Cobb and Douglas [1] set forth a pathbreaking theory of aggregate manufacturing production, based on a highly tractable functional form of production function. Through the decades, such a function has proved extremely relevant for both production and consumption analysis, so as to become "perhaps the most ubiquitous function in all of economics." [2].

Being homothetic, Cobb-Douglas (CD) functions embody the scale symmetry of production and consumption problems, which has been long recognized as a benchmark property with noticeable implications (in first instance, the factorization of expenditure functions). In a recent paper, Mantovi [3] deepens the benchmark nature of homothetic models in terms of the commutativity of expansion and substitution effects. True, by their scale invariance, homothetic functions cannot represent general traits of preferences.

Definitely, to some extent, the luxury-necessity dichotomy can accommodate general traits of preferences. Departing from the scale symmetry of homothetic expansion paths (rays), the luxury-necessity dichotomy posits that, roughly speaking, expansion paths bend monotonically towards luxury. It is the aim of the present contribution to discuss a class of transcendental preferences for luxury which enable us to solve explicitly the consumption problem, maintain the pleasant analytical tractability of CD models, and then introduce a differential measure of luxury which complements and improves upon income elasticity of demand (IED).

The plan of the rest of the paper is as follows. In section 2 we introduce our preferences, solve the consumption problem, and represent explicitly income elasticity of demand. In section 3 we introduce our measure of luxury, and discuss its connection with Shephard's distance. A final section sketches potential lines of progress.

II. A Model of Luxury-Necessity Dichotomy

A. *Transcendental preferences for luxury*

Consider the 2-parameter class of preferences represented by the ordinal utility functions

$$U_{a,\varepsilon}(x,y) = x^a (ye^{\varepsilon y})^{1-a} \tag{1}$$

Such functions belong to the class of *transcendental* functions [6], and then to the class of *generalized power* production functions [7]. The class (1) is parametrized by the parameter $a \in (0,1)$, and by the "luxury" parameter $\varepsilon \in [0,\infty)$, which 'injects' increasing luxury into good *y*; generalizes the corresponding CD parameter in that (1) approach the CD form for $\varepsilon \to 0$.

Indifference curves for the class (1) read

$$x(y;u) = u^{\frac{1}{a}} (ye^{\varepsilon y})^{\frac{a-1}{a}} \tag{2}$$

Sample curves (2) are represented in Figure 1.

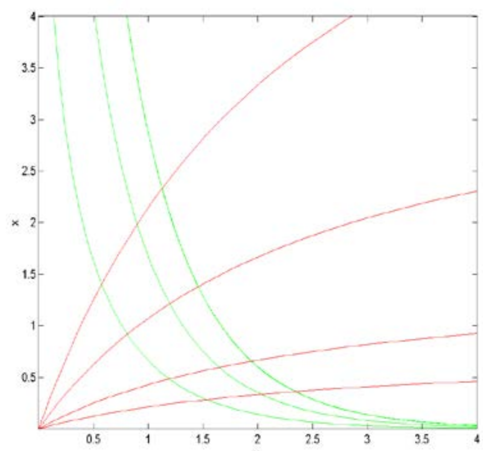

Figure 1. Sample indifference curves (utility levels 2, 4, 6) for the agent (a,ε) = (0.75, 0.4) against expansion paths p_x/p_y = 1, 2, 5, 10.

As expected, utility curves approach the corresponding CD curves in the "no-luxury" limit $y \to 0$.

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B. Marshallian demand

By the analytical tractability of preferences (1), we are in a position to solve the consumption problem with only minor deviations from the analytics of CD models. The FOC

$$MRS = \frac{\frac{\partial U}{\partial y}}{\frac{\partial U}{\partial x}} = \frac{p_y}{p_x} = \frac{1-a}{a}\frac{x}{y}(1+\varepsilon y)$$
$$\Rightarrow \; x p_x = \frac{a}{1-a}\frac{y p_y}{1+\varepsilon y} \tag{3}$$

is a smooth generalization of the corresponding CD condition, (a condition, evidently, independent of the utility representation), and provides a Cartesian equation for expansion paths.

Marshallian demand functions satisfy the budget balance condition; therefore, assume all income is spent, plug (3) into the budget constraint and obtain the simple quadratic equation

$$\varepsilon\, y^2 + \left(\frac{1}{1-a} - \varepsilon\frac{I}{p_y}\right) y - \frac{I}{p_y} = 0 \tag{4}$$

whose positive solution

$$y_{1,2} = \frac{1}{2\varepsilon}\left(-\frac{1}{1-a} + \varepsilon\frac{I}{p_y} \pm \sqrt{\left(\frac{1}{1-a} - \varepsilon\frac{I}{p_y}\right)^2 + 4\varepsilon\frac{I}{p_y}}\right) \tag{5}$$

establishes the Marshallian demand of the luxury good for the agent (a,ε). Notice, and enter such an expression via their ratio, so as to guarantee homogeneity of degree 0.

The simplicity of the function (5), a combination of elementary functions, enables us to plot Engle curves for both the goods with great analytical control. Evidently, given (5), the Marshallian demand for the necessary good is uniquely determined by the budget constraint. Consistently with the cartesian representation of expansion paths (eq. 3), the necessary good is subject to satiation: for any pair p_x, p_y, the upper bound $\bar{x} = \frac{a}{1-a}\frac{p_y}{p_x}\frac{1}{\varepsilon}$ for the consumption of the necessary good is the satiation level, which, as expected, varies with relative price, and shrinks for increasing ε.

On the other hand, the consumption of the luxury good, as income increases, “takes it all”: for large enough income, the necessity share becomes negligible, and Engel curves for luxury are linear. Figure 2 provides a transparent representation of such a simple pattern, which witnesses the effectiveness of our consumption model in representing fundamental traits of behavior.

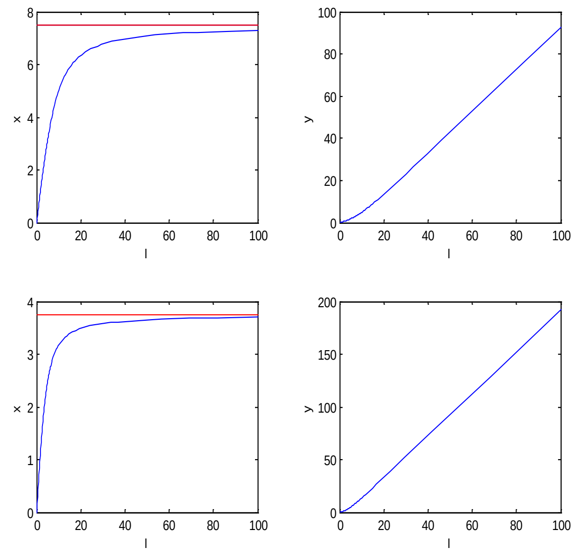

Figure 2. Engel curves for the agent (a,ε) = (0.75, 0.4) for unit prices (above) and p_x=2p_y (below) for both luxury (right) and necessary (left) good.

C. Income elasticity of demand

IED exceeding 1 is the standard indicator of a good being a luxury good (see for instance [7] for a landmark theoretical/empirical analysis). By the analytical tractability of our model we are in a position to write simple explicit solutions for such an indicator. For the sake of definiteness, consider unit prices, and then optimal consumption as a function of the expenditure $I = x + y$. Then, equation (4) reduces to $0 = y^2 + (10-I)y - 2.5I$, whose positive solution reads

$$y(I) = \frac{1}{2}\left(I - 10 + \sqrt{(I-10)^2 + 10I}\right) \tag{6}$$

The graph of (6) is the Engel curve for the luxury commodity at identical prices for the agent . Differentiate (6) and divide by itself and obtain the expression

$$\frac{I}{y^*}\frac{dy^*}{dI} = I\,\frac{1 + \frac{I-10+5}{\sqrt{(I-10)^2+10I}}}{I - 10 + \sqrt{(I-10)^2 + 10I}} \tag{7}$$

for IED as a function of expenditure (income) I. The plot of (7) function is given in Figure 3.

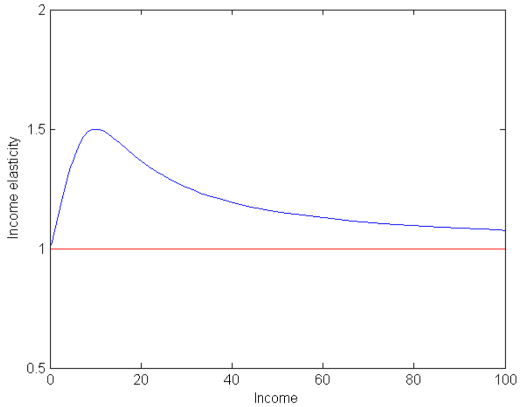

Figure 3. Income elasticity of demand as a function of income for the agent $(a,\varepsilon) = (0.75, 0.4)$.

As expected, such a plot approaches 1 for vanishing expenditure (in which preferences approach the CD limit), display an initial phase of growth (driven by the bending of the expansion path) which culminates in a peak, after which the function approaches asymptotically 1 from above (as the expansion path approach the asymptotic satiation of necessity.

The transparency of such a picture witnesses the effectiveness of our model of preferences. True, the curve represented in Figure 2 spans a vertical range [1,1.5], and the question raises naturally as to which of such values should be considered a 'preferred' indicator of the luxury effect driving the agents under inspection. Evidently, if, ceteris paribus, we increase ε, we obtain a curve with the same qualitative behavior and with a magnification in the vertical direction.

In fact, IED is a function of dual variables (prices and income). Building on the philosophy represented in [3, 4] we can define a primal indicator representing a 'departure' from the scale symmetry (homotheticity) of CD models which bears a close connection with Shephard' distance.

III. Measuring luxury via Lie brackets

Along the line of though discussed by [3] one can employ vector fields on consumption space the model globally relevant economic effects, and then employ Lie brackets as measures of noncommutativity of such effects. We already know ([3]) that in the CD limit $\varepsilon \to 0$ our agents display commutation of expansion and substitution effects. In addition, we expect noncomutativity to onset for positive ε, and to increase with such a parameter.

The vector field on our consumption set generating scale effects (SCE) reads [3,4]:

$$\mathbf{Z} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{8}$$

It is a radial vector field whose components do coincide with the coordinates of the base, and such that, for any function f homogeneous of degree d, Euler's theorem can be written $\mathbf{Z}(f) = d\, f$. Recall, scale transformation define the rationale for Shephard's distance [9], which provides a primal representation of preferences equivalent to the one given by a utility function.

As of the vector field generating substitution effects (SUE), let us follow [3], and notice that the vector field

$$\begin{aligned}\tilde{S} &= \frac{\partial U}{\partial y}\frac{\partial}{\partial x} - \frac{\partial U}{\partial x}\frac{\partial}{\partial y} \\ &= e^{\varepsilon(1-a)y}\left((1-a)x^{a}y^{-a}(1+\varepsilon y)\frac{\partial}{\partial x} - ax^{a-1}y^{1-a}\frac{\partial}{\partial y}\right)\end{aligned} \tag{9}$$

is tangent to indifference curves, so that

Let the ratio y/x be the coordinate on indifference curves by means of which we want to parametrize SUE; such a choice is pivotal for the consistency of our framework, in that such a parametrization of SUE is *adapted* to SCE. Compute the normalization function, i.e. the action of the vector field (23) on the function y/x,

$$\begin{aligned}\mathcal{N}_{\frac{y}{x}} &= \tilde{S}\left(\frac{y}{x}\right) = \left(\frac{\partial U}{\partial y}\frac{\partial}{\partial x} - \frac{\partial U}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{y}{x}\right) \\ &= -e^{\varepsilon(1-a)y}x^{a-2}y^{1-a}(1+(1-a)\varepsilon y) = -\frac{1}{x^{2}}\chi_{a,\varepsilon}(y)U_{a,\varepsilon}(x,y)\end{aligned} \tag{10}$$

As expected, we face the emergence of the deviation factor χ. Thus, the proper SUE vector field results in

$$S \equiv \frac{\tilde{S}}{\mathcal{N}_{\frac{y}{x}}} == -(1-a)\,x^{2}y^{-1}\frac{1+\varepsilon y}{\chi_{a,\varepsilon}(y)}\frac{\partial}{\partial x} + \frac{ax}{\chi_{a,\varepsilon}(y)}\frac{\partial}{\partial y} \tag{11}$$

so that $S\left(\frac{y}{x}\right) = 1$, with the correct CD limit for $\varepsilon \to 0$ (Mantovi, 2013a). The vector field (25) is clearly independent of the utility representation,

Thus, we are in a position to compute the Lie bracket between the substitution vector field (11) and scaling vector field (8) by means of the standard algebra [10, p. 153] and obtain

$$
\begin{aligned}
\left[\boldsymbol{s},\mathbf{Z}\right]_x &= \boldsymbol{s}_x \frac{\partial Z_x}{\partial x} + \boldsymbol{s}_y \frac{\partial Z_x}{\partial y} - Z_x \frac{\partial \boldsymbol{s}_x}{\partial x} - Z_y \frac{\boldsymbol{s}_x}{\partial y} \\
&= -(1-a)x^2 y^{-1} \frac{1+\varepsilon y}{\chi(y)} \cdot 1 + ax \frac{1}{\chi(y)} \cdot 0 \\
&\quad - x(a-1)2xy^{-1} \frac{1+\varepsilon y}{\chi(y)} - y(1-a)x^2 y^{-2} \frac{1+\varepsilon y}{\chi(y)} \\
&\quad + y(1-a)x^2 y^{-1} \frac{d}{dy}\left(\frac{1+\varepsilon y}{\chi(y)}\right) \\
&= (1-a)x^2 \frac{d}{dy}\left(\frac{1+\varepsilon y}{\chi(y)}\right)
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\left[\boldsymbol{s},\mathbf{Z}\right]_y &= \boldsymbol{s}_x \frac{\partial Z_y}{\partial x} + \boldsymbol{s}_y \frac{\partial Z_y}{\partial y} - Z_x \frac{\partial \boldsymbol{s}_y}{\partial x} - Z_y \frac{\partial \boldsymbol{s}_y}{\partial y} \\
&= -(1-a)x^2 y^{-1} \frac{1+\varepsilon y}{\chi(y)} \cdot 0 + ax \frac{1}{\chi(y)} \cdot 1 \\
&\quad - xa \frac{1}{\chi(y)} - y \cdot 0 \frac{1}{\chi(y)} - yax \frac{d}{dy} \frac{1}{\chi(y)} \\
&= -yax \frac{d}{dy} \frac{1}{\chi(y)}
\end{aligned} \tag{13}
$$

We thereby verify that the vector field with components (12) and (13) is radial, on account of the simple derivatives

$$
\begin{aligned}
\frac{d}{dy}\left(\frac{1+\varepsilon y}{\chi(y)}\right) &= \frac{a\varepsilon}{(1+(1-a)\varepsilon y)^2} \\
\frac{d}{dy} \frac{1}{\chi(y)} &= \frac{(a-1)\varepsilon}{(1+(1-a)\varepsilon y)^2}
\end{aligned} \tag{14}
$$

To sum up, the vector field with components (12) and (13) is a consistent measure of deviation from scale symmetry (homotheticity), in which scale effects and substitution effects do commute [3].

$$
\begin{aligned}
[\boldsymbol{s},\mathbf{Z}] &= \varepsilon \frac{a(1-a)}{(1+(1-a)\varepsilon y)^2}\left(x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}\right) \\
&= \varepsilon\, a(1-a) \frac{x}{\chi_{a,\varepsilon}{}^2(y)} \mathrm{Z} \equiv L(x,y;a,\varepsilon)\, \mathrm{Z}
\end{aligned} \tag{15}
$$

Such a vector field, as expected, is radial, in that the failure of the infinitesimal path employed in standard discussions [10 Spivak] to close up takes place in the radial direction, and is uniform in *x*/*y*. Figure 4 provides an intuitive setting with respect to which to pin down the insight connecting (15) with Shephard's distance.

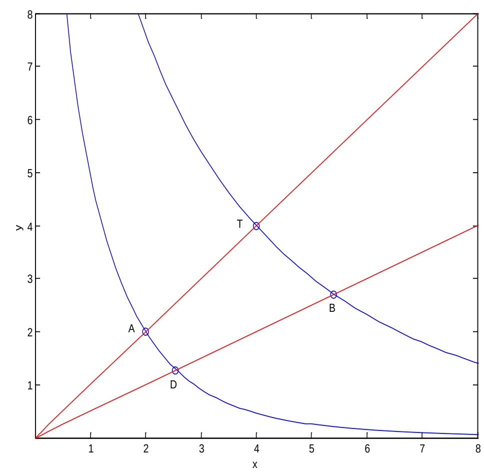

Figure 4. The noncommutativity of finite scale effects and substitution effects: the loop ATBD is closed since AT and DB represent different scale effects.

As represented in Figure 4, closed loops must entail different scale effects: the scale effect connecting D and B is larger than the scale effect connecting A with T since such scale effects connect point on a higher indifference curve with points on different indifference curves, we are given a monotone pattern of Shephard's distances between the points on the upper curve and the lower utility level. Such measures are the finite correspondent of the infinitesimal measure (15), which defines an "index" of luxury as a function of primal variables and of the parameters of the model. We thereby obtain a natural representation of the connection between Shephard's distance and the commutativity of SCE and SUE

IV. Perspectives

The "index" of luxury we been arguing about, evidently, is shaped by the properties of the specific functional form (1). Still, the degree of generality embodied by our model, in which the bending of expansion path increases with ε, enables us to believe in the 'universality' of the insights thereby conveyed.

On the one hand, by their analytical tractability, the preferences (1) seem to represent a promising building block for theoretical general equilibrium analysis. On the other hand, the well behavior of the consumption pattern discussed leads one to conjecture potential applications to the positive theory of general equilibrium. On the other hand, the relevance of potential empirical applications may rest on the smooth Cobb-Douglas limit of our model.

Overall productive efficiency as tailored by [5] seems to be a natural playground for our approach, on account of the close link established in [3] between, on the one hand, standard and reversed Farrel decompositions, and, on the other hand, the commutativity of expansion and substitution effects.

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