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Some results via de la Vallée-Poussin mean in probabilistic 2-normed spaces

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Abstract—In this article we introduce some new type of summability methods for double sequences involving the ideas of de la Vallée-Poussin mean in probabilistic 2 -normed space and examine some important results.

Keywords—t -norm, probabilistic 2 -normed space, double sequence, statistical completeness, de la Vallé e-Poussin mean.

I. INTRODUCTION

PROBABILISTIC normed space is significant as a generalization of deterministic results of linear normed spaces. In a PN space, the norms of the vectors are represented by probability distribution functions instead of nonnegative real numbers. If x is an element of a PN space, then its norm is denoted by F_x, and the value $F_x(t)$ is interpreted as the probability that the norm of x is smaller than t. In [24], probabilistic normed spaces were first introduced by Śerstnev and then by it was extended to random/probabilistic 2 -normed spaces by Golet [5] using the notion of 2 -norm which is defined by Gähler [3,4] and since then, many researchers have studied these subjects and obtained various results [6-8,23,27,28]. Afterwards, Alsina et al. [1] presented a new definition of a PN space which includes the definition of Śerstnev [25] as a special case. This new definition rapidly became the standard one and it has been adopted by many authors (for instance, [9-16,19,20]).

The concepts of statistical convergence for sequences of real numbers was introduced (independently) by Steinhaus [26] and Fast [2]. The concept of statistical convergence was further discussed and developed by many authors in more general abstract spaces [6,9-11,13,20].

Some new type of summability methods for double sequences involving the ideas of de la Vallée-Poussin mean has not been studied previously in the setting of probabilistic 2 -normed (PTN) spaces. Motivated by this fact, in this paper, the notion of (λ,μ)-summable, statistically (λ,μ)-summable, statistically (λ,μ)-Cauchy and statistically (λ,μ)-complete for double sequence with respect to PTN-space and establish some interesting results.

II. DEFINITIONS AND NOTATIONS

First we recall some of the basic concepts, which will be used in this paper.

The notion of convergence for double sequence was introduced by Pringsheim [18]: We say that a double sequence $x=(x_{j,k})_{j,k\in\mathbb{N}}$ of reals is convergent to L in Pringsheim's sense (shortly, (P) convergent) provided that $\varepsilon>0$ there exists a positive integer N such that $|x_{j,k}-L|<\varepsilon$ whenever $j,k\geq N$.

Statistical convergence for double sequences $x=(x_{jk})$ of real numbers was introduced and studied by Mursaleen and Edely [17] as follows: Let $K\subseteq\mathbb{N}\times\mathbb{N}$ and $K(h,l)=\{j\leq h,k\leq l\ :\ (j,k)\in A\}$, where $h,l\in\mathbb{N}$. Then we define upper and lower asymptotic density of a two-dimensional set K, respectively

$$\overline{\delta}_2(K):=(P)\limsup_{h,l\to\infty}\frac{|K(h,l)|}{hl};\ \underline{\delta}_2(K):=(P)\liminf_{h,l\to\infty}\frac{|K(h,l)|}{hl}.$$

If $\overline{\delta}_2(K)=\underline{\delta}_2(K)$, then the common value $\delta_2(K)$ is called the double asymptotic density of the set K and

$$\delta_2(K)=(P)\lim_{h,l\to\infty}\frac{|K(h,l)|}{hl}.$$

The double sequence $x=(x_{j,k})$ statistically converges to a point L if for each $\varepsilon>0$ we have $\delta_2(K(\varepsilon))=0$, where $K(\varepsilon)=\{(j,k), j\leq h,k\leq l\ :\ |x_{j,k}-L|\geq\varepsilon\}$ and in such situation we will write $L=st\text{-}\lim x$ (or $x_{j,k}\to L(st)$)

Let $\lambda=(\lambda_m)$ and $\mu=(\mu_n)$ are two non-decreasing sequences of positive numbers tending to ∞ such that

$$\lambda_{m+1}\leq\lambda_m+1,\ \lambda_1=0 \text{ and } \mu_{n+1}\leq\mu_n+1,\ \mu_1=0.$$

Recall that (λ,μ)-density of the set $K \subseteq \mathbb{N}\times\mathbb{N}$ is given by

$$\delta_{\lambda,\mu}(K) = (P)\lim_{m,n} \frac{1}{\lambda_m\mu_n}\Big|\{m-\lambda_m+1 \le j \le m,\ n-\mu_n+1 \le k \le n \ : \ (j,k)\in K\}\Big|$$

provided that the limit exists. If $\lambda_m = m$ for all m, and $\mu_n = n$ for all n, the (λ,μ)-density is reduced to the double natural density.

The generalized double de la Valée-Pousin mean is defined by

$$t_{m,n}(x) = \frac{1}{\lambda_m\mu_n}\sum_{(j,k)\in J_m\times I_n} x_{jk},$$

where $J_m = [m-\lambda_m+1, m]$ and $I_n = [n-\mu_n+1, n]$.

We say that $x = (x_{jk})$ is (λ,μ)-statistically convergent to the number L if for every $\varepsilon > 0$,

$$(P)\lim_{m,n}\frac{1}{\lambda_m\mu_n}\Big|\{j\in J_m, k\in I_n \ : \ |x_{jk}-L| \ge \varepsilon\}\Big| = 0.$$

and in such situation we will write $st_{\lambda,\mu}\text{-}\lim x = L$.

Definition 1. *([3,4]) Let X be a real vector space of dimension d, where $2 \le d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X\times X \to \mathbb{R}$ which satisfies* **(i)** $\|x,y\| = 0$ *if and only if x and y are linearly dependent;* **(ii)** $\|x,y\| = \|y,x\|$; **(iii)** $\|\alpha x,y\| = |\alpha|\,\|x,y\|$, $\alpha\in\mathbb{R}$; **(iv)** $\|x,y+z\| \le \|x,y\| + \|x,z\|$. *The pair $(X,\|\cdot,\cdot\|)$ is then called a 2-normed space.*

As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x,y\| :=$ the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x,y\| = |x_1y_2 - x_2y_1|, \quad x = (x_1,x_2), \quad y = (y_1,y_2).$$

Observe that in any 2-normed space $(X,\|\cdot,\cdot\|)$ we have $\|x,y\| \ge 0$ and $\|x,y+\alpha x\| = \|x,y\|$ for all $x,y\in X$ and $\alpha\in\mathbb{R}$. Also, if x, y and z are linearly dependent, then $\|x,y+z\| = \|x,y\| + \|x,z\|$ or $\|x,y-z\| = \|x,y\| + \|x,z\|$. Given a 2-normed space $(X,\|\cdot,\cdot\|)$, one can derive a topology for it via the following definition of the limit of a sequence: a sequence (x_n) in X is said to be convergent to x in X if $\lim_{n\to\infty}\|x_n - x, y\| = 0$ for every $y\in X$.

All the concepts listed below are studied in depth in the fundamental book by Schweizer and Sklar [22].

Definition 2. *Let $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = \{x\in\mathbb{R} : x \ge 0\}$ and $S = [0,1]$ the closed unit interval. A mapping $f : \mathbb{R}\to S$ is called a distribution function if it is nondecreasing and left continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$.*

We denote the set of all distribution functions by D^+ such that $f(0) = 0$. If $a\in\mathbb{R}_+$, then $H_a\in D^+$, where

$$H_a(t) = \begin{cases} 1, & \text{if } t > a, \\ 0, & \text{if } t \le a. \end{cases}$$

It is obvious that $H_0 \ge f$ for all $f\in D^+$.

Definition 3. *A triangular norm (t-norm) is a continuous mapping $*: S\times S\to S$ such that $(S,*)$ is an abelian monoid with unit one and $c*d \le a*b$ if $c \le a$ and $d \le b$ for all $a,b,c,d\in S$. A triangle function τ is a binary operation on D^+ which is commutive, associative and $\tau(f,H_0) = f$ for every $f\in D^+$.*

Definition 4. *Let X be a linear space of dimension greater than one, τ is a triangle, and $\mathbf{F} : X\times X\to D^+$. Then $\mathbf{F}$ is called a probabilistic 2-norm and $(X,\mathbf{F},\tau)$ a probabilistic 2-normed space if the following conditions are satisfied:*

(2.2.1) $\mathbf{F}(x,y;t) = H_0(t)$ *if x and y are linearly dependent, where $\mathbf{F}(x,y;t)$ denotes the value of $\mathbf{F}(x,y)$ at $t\in\mathbb{R}$,*

(2.2.2) $\mathbf{F}(x,y;t) \ne H_0(t)$ *if x and y are linearly independent,*

(2.2.3) $\mathbf{F}(x,y;t) = \mathbf{F}(y,x;t)$ *for all $x,y\in X$,*

(2.2.4) $\mathbf{F}(\alpha x,y;t) = \mathbf{F}(x,y;\frac{t}{|\alpha|})$ *for every $t>0, \alpha\ne 0$ and $x,y\in X$,*

(2.2.5) $\mathbf{F}(x+y,z;t) \ge \tau(\mathbf{F}(x,z;t),\mathbf{F}(y,z;t))$ *whenever $x,y,z\in X$.*

III. Main results

In this section, our aim is to define some concepts of (λ,μ)-summable, statistically (λ,μ)-summable, statistically (λ,μ)-Cauchy and statistically (λ,μ)-complete for double with respect to PTN-space and obtain some interesting results.

Definition 5. *Let $(X,\mathbf{F},\tau)$ be a PTN space. The double sequence $x = (x_{jk})$ in X is said to be (λ,μ)-summable (or briefly, $\mathbf{F}(\lambda,\mu)$-summable) to L if for each $\varepsilon>0$, $q\in(0,1)$ and each nonzero $z\in X$ there exists $N\in\mathbb{N}$ such that $\mathbf{F}(t_{m,n}(x)-L,z;\varepsilon) > 1-q$ for all $m,n>N$. In this case, we write $\mathbf{F}(\lambda,\mu)\text{-}\lim x, z = L$.*

Definition 6. *Let* $(X,\mathbf{F},\tau)$ *be a PTN space. The double sequence* $x=(x_{jk})$ *in* X *is said to be statistically* (λ,μ)*-summable (or briefly,* $\mathbf{F}(st_{\lambda,\mu})$*-summable) to* L *if* $\delta_2(K_{\lambda,\mu})=0,$ *where*

$$K_{\lambda,\mu}=\{(m,n)\in\mathbb{N}\times\mathbb{N} : \mathbf{F}(t_{m,n}(x)-L,z;\varepsilon)\le 1-q\},$$

i.e. if for each $\varepsilon>0,$ $q\in(0,1)$ *and each nonzero* $z\in X$

$$(P)\lim_{h,l}\frac{1}{hl}\left|\{m\le h,n\le l : \mathbf{F}(t_{m,n}(x)-L,z;\varepsilon)\le 1-\lambda\}\right|=0$$

or, equivalently

$$(P)\lim_{h,l}\frac{1}{hl}\left|\{m\le h,n\le l : \mathbf{F}(t_{m,n}(x)-L,z;\varepsilon)>1-\lambda\}\right|=1.$$

In this case, we write $\mathbf{F}(st_{\lambda,\mu})$ - $\lim x,z=L,$ *and* L *is called* $\mathbf{F}(st_{\lambda,\mu})$*-limit of* x*.*

Definition 7. *Let* $(X,\mathbf{F},\tau)$ *be a PTN space. The double sequence* $x=(x_{jk})$ *in* X *is said to be statistically* (λ,μ)*-Cauchy (or briefly,* $\mathbf{F}(st_{\lambda,\mu})$*-Cauchy) if for each* $\varepsilon>0,$ $q\in(0,1)$ *and each nonzero* $z\in X$ *there exists* $M,N\in\mathbb{N}$ *such that for all* $m,p\ge M,$ $n,q\ge N,$ *the set* $S_\varepsilon(\lambda,\mu)=\{(m,n)\in\mathbb{N}\times\mathbb{N} : \mathbf{F}(t_{m,n}(x)-t_{p,q}(x),z;\varepsilon)\le 1-q\}$ *has double natural density zero, i.e.*

$$(P)\lim_{h,l}\frac{1}{hl}\left|\{m\le h,n\le l : \mathbf{F}(t_{m,n}(x)-t_{p,q}(x),z;\varepsilon)\le 1-q\}\right|=0.$$

Theorem 1. *Let* $(X,\mathbf{F},\tau)$ *be a PTN space. If a double sequence* $x=(x_{jk})$ *in* X *is statistically* (λ,μ)*-summable, that is,* $\mathbf{F}(st_{\lambda,\mu})$ - $\lim x,z=L$ *exists, then* $\mathbf{F}(st_{\lambda,\mu})$ - $\lim x,z$ *is unique.*

Proof. Suppose that $\mathbf{F}(st_{\lambda,\mu})$ - $\lim x,z=L_1$ and $\mathbf{F}(st_{\lambda,\mu})$ - $\lim x,z=L_2,$ where $L_1\ne L_2.$ For given $\varepsilon>0$ and each nonzero $z\in X,$ select $q>0$ such that $\tau((1-q),(1-q))>1-\varepsilon.$ Then, for any $t>0$, we define

$$A_q(\lambda,\mu)=\{(m,n)\in\mathbb{N}\times\mathbb{N} : \mathbf{F}(t_{m,n}(x)-L_1,z;t)\le 1-q\}$$

and

$$B_q(\lambda,\mu)=\{(m,n)\in\mathbb{N}\times\mathbb{N} : \mathbf{F}(t_{m,n}(x)-L_2,z;t)\le 1-q\}.$$

Since, $\mathbf{F}(st_{\lambda,\mu})$ - $\lim x,z=L_1$ implies $\delta_2(A_\lambda(\lambda,\mu))=0$ and similarly, we have $\delta_2(B_\lambda(\lambda,\mu))=0.$ Now, let $C_q(\lambda,\mu)=A_q(\lambda,\mu)\cap B_q(\lambda,\mu).$ It follows that $\delta_2(C_q(\lambda,\mu))=0$ and hence the complement $C_q^c(\lambda,\mu)$ is nonempty set and $\delta_2(C_q^c(\lambda,\mu))=1.$ Now, if $(m,n)\in\mathbb{N}\times\mathbb{N}\setminus C_q(\lambda,\mu),$ then

$$\mathbf{F}(L_1-L_2,z;t)\ge\tau\left(\mathbf{F}\left(t_{m,n}(x)-L_1,z;\frac{t}{2}\right),\mathbf{F}\left(t_{m,n}(x)-L_2,z;\frac{t}{2}\right)\right)$$
$$>\tau((1-q),(1-q))>1-\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we get $\mathbf{F}(L_1-L_2,z;t)=1$ for all $t>0$ and each nonzero $z\in X.$ Hence $L_1=L_2,$ which proves theorem.

Theorem 2. *Let* $(X,\mathbf{F},\tau)$ *be a PTN space. If a double sequence* $x=(x_{jk})$ *in* X *is* $\mathbf{F}(\lambda,\mu)$*-summable to* $L,$ *then it is* $\mathbf{F}(st_{\lambda,\mu})$*-summable to the same limit.*

Proof. Let us consider that $\mathbf{F}(\lambda,\mu)$ - $\lim x,z=L.$ For every $\varepsilon>0,t>0$ and nonzero $z\in X,$ there exists positive integer N such that

$$\mathbf{F}(t_{m,n}(x)-L,z;t)>1-\varepsilon$$

holds for all $m,n\ge N.$ Since

$$M_\varepsilon(\lambda,\mu)=\{(m,n)\in\mathbb{N}\times\mathbb{N} : \mathbf{F}(t_{m,n}(x)-L,z;t)\le 1-\varepsilon\}$$

is contained in $\mathbb{N}\times\mathbb{N}$. Therefore $\delta_2(M_\varepsilon(\lambda,\mu))=0,$ that is, $x=(x_{jk})$ is $\mathbf{F}(st_{\lambda,\mu})$-summable to $L.$

The following example shows that the converse of Theorem 2 need not be true.

Example 1. *Consider* $X=\mathbb{R}^2$ *with* $\|x,y\|:=|x_1y_2-x_2y_1|$ *where* $x=(x_1,x_2),$ $y=(y_1,y_2)\in\mathbb{R}^2$ *and let* $\tau(a,b)=ab$ *for all* $a,b\in S.$ *For all* $(x,y)\in\mathbb{R}^2$ *and* $t>0,$ *consider*

$$\mathbf{F}_{x,y}(t)=\frac{t}{t+\|x,y\|}.$$

Then $(\mathbb{R}^2,\mathbf{F},\tau)$ *is a PTN space. The double sequence* $x=(x_{jk})$ *is defined by*

$$t_{m,n}(x)=\begin{cases} mn, & \text{if } m,n=w^2,\ w\in\mathbb{N}\\ 0, & \text{otherwise.}\end{cases}$$

For $\varepsilon>0,t>0$ *and nonzero* $z\in X,$ *write*

$$M_\varepsilon(\lambda,\mu)=\{(m,n)\in\mathbb{N}\times\mathbb{N} : \mathbf{F}(t_{m,n}(x),z;t)\le 1-\varepsilon\}.$$

It is easy to see that

$$\mathbf{F}(t_{m,n}(x),z;t)=\frac{t}{t+\|t_{m,n}(x),y\|}=\begin{cases}\frac{t}{t+mnz}, & \text{for } m,n=w^2,\ w\in\mathbb{N}\\ 1, & \text{otherwise;}\end{cases}$$

and hence

$$\lim\mathbf{F}(t_{m,n}(x),z;t)=\begin{cases}0, & \text{if } m,n=w^2,\ w\in\mathbb{N}\\ 1, & \text{otherwise.}\end{cases}.$$

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We see that the sequence $x=(x_{jk})$ *is not* $\mathsf{F}(\lambda,\mu)$*-summable in* $(\mathsf{R}^2,\mathsf{F},\tau)$*. But the set* $M_\varepsilon(\lambda,\mu)$ *has double natural density zero since* $M_\varepsilon(\lambda,\mu)\subset\{(1,1),(4,4),(9,9),...\}$*. From here, we obtain that the converse of Theorem 2 need not be true.*

Theorem 3. *Let* (X,F,τ) *be a PTN space. If a double sequence* $x=(x_{jk})$ *in* X *is statistically* $\mathsf{F}(st_{\lambda,\mu})$*-summable to* L *if and only if there exists a subset*

$$M=\{(j_m,k_n):\ j_1<j_2<...;\ k_1<k_2<...\}\subseteq\mathsf{N}\times\mathsf{N}$$

such that $\delta_2(K)=1$ *and* $\mathsf{F}(\lambda,\mu)\text{-}\lim x_{j_m,k_n}, z=L.$

Proof. Suppose that there exists a subset

$$M=\{(j_m,k_n):\ j_1<j_2<...;\ k_1<k_2<...\}\subseteq\mathsf{N}\times\mathsf{N}$$

such that $\delta_2(M)=1$ and $\mathsf{F}(\lambda,\mu)\text{-}\lim x_{j_m,k_n}, z=L.$ Then there exists $N\in\mathsf{N}$ such that $\mathsf{F}(t_{m,n}(x)-L,z;t)>1-\varepsilon$ holds for all $m,n>N$. Put

$$M_\varepsilon(\lambda,\mu)=\{(m,n)\in\mathsf{N}\times\mathsf{N}\ :\ \mathsf{F}(t_{j_m,k_n}(x)-L,z;t)\le 1-\varepsilon\}$$

and $M'=\{(j_{N+1},k_{N+1}),(j_{N+2},k_{N+2}),...\}$. Then $\delta_2(M')=1$ and $M_\varepsilon(\lambda,\mu)\subseteq\mathsf{N}-K'$ which implies that $\delta_2(M_\varepsilon(\lambda,\mu))=0.$ Hence $x=(x_{jk})$ is statistically (λ,μ)-summable to L in PTN space.

Conversely, suppose that $x=(x_{jk})$ is $\mathsf{F}(st_{\lambda,\mu})$-summable to L. For $q=1,2,...$ and $t>0,$ write

$$M_q(\lambda,\mu)=\left\{(m,n)\in\mathsf{N}\times\mathsf{N}\ :\ \mathsf{F}(t_{j_m,k_n}(x)-L,z;t)\le 1-\frac{1}{q}\right\},$$

and

$$K_q(\lambda,\mu)=\left\{(m,n)\in\mathsf{N}\times\mathsf{N}\ :\ \mathsf{F}(t_{j_m,k_n}(x)-L,z;t)>\frac{1}{q}\right\}.$$

Then $\delta_2(M_q(\lambda,\mu))=0$ and

$$K_1(\lambda,\mu)\supset K_2(\lambda,\mu)\supset ...K_i(\lambda,\mu)\supset K_{i+1}(\lambda,\mu)\supset ... \tag{1}$$

and

$$\delta_2(K_q(\lambda,\mu))=1,\ q=1,2,.... \tag{2}$$

Now, we have to show that $(m,n)\in K_q(\lambda,\mu)$, $x=(x_{j_m,k_n})$ is $\mathsf{F}(\lambda,\mu)$-summable to L. Assume that $x=(x_{j_m,k_n})$ is not $\mathsf{F}(\lambda,\mu)$-summable to L. Hence, there exists $\varepsilon>0$ such that $\mathsf{F}(t_{j_m,k_n}(x)-L,z;t)\le\varepsilon$ for infinitely many terms. Let

$$K_\varepsilon(\lambda,\mu)=\{(m,n)\in\mathsf{N}\times\mathsf{N}\ :\ \mathsf{F}(t_{j_m,k_n}(x)-L,z;t)>\varepsilon\},$$

and $\varepsilon>\frac{1}{q}$ with $q=1,2,3,....$ Then

$$\delta_2(K_\varepsilon(\lambda,\mu))=0,$$

and by (1), $K_q(\lambda,\mu)\subset K_\varepsilon(\lambda,\mu)$. Hence $\delta_2(K_q(\lambda,\mu))=0,$ which contradicts (2) and therefore $x=(x_{j_m,k_n})$ is $\mathsf{F}(\lambda,\mu)$-summable to L.

Theorem 4. *Let* (X,F,τ) *be a PTN space. If a double sequence* $x=(x_{jk})$ *in* X *is* (λ,μ)*-summable, then it is statistically* (λ,μ)*-Cauchy.*

Proof. Assume that $\mathsf{F}(st_{\lambda,\mu})\text{-}\lim x, z=L.$ Let $\varepsilon>0$ be a given number so that choose $q>0$ such that

$$\tau((1-q),(1-q))>1-\varepsilon.$$

Then, for $t>0$ and nonzero $z\in X$, we have

$$\delta_2(A_q(\lambda,\mu))=0, \tag{3}$$

where $A_q(\lambda,\mu)=\{(m,n)\in\mathsf{N}\times\mathsf{N}\ :\ \mathsf{F}(t_{m,n}(x)-L,z;\frac{t}{2})\le 1-q\}$ which implies that

$$\delta_2(A_q^c(\lambda,\mu))=\delta_2\left(\left\{(m,n)\in\mathsf{N}\times\mathsf{N}\ :\ \mathsf{F}\left(t_{m,n}(x)-L,z;\frac{t}{2}\right)>1-q\right\}\right)=1.$$

Let $(f,g)\in A_q^c(\lambda,\mu)$. Then $\mathsf{F}(t_{f,g}(x)-L,z;\frac{t}{2})>1-q.$ Now, let

$$B_\varepsilon(\lambda,\mu)=\left\{(m,n)\in\mathsf{N}\times\mathsf{N}\ :\ \mathsf{F}\left(t_{m,n}(x)-t_{f,g}(x),z;\frac{t}{2}\right)\le 1-\varepsilon\right\}.$$

We need to show that $B_\varepsilon(\lambda,\mu)\subset A_q(\lambda,\mu)$. Let $(m,n)\in B_\varepsilon(\lambda,\mu)\setminus A_q(\lambda,\mu)$. Then

$\mathsf{F}(t_{m,n}(x)-t_{f,g}(x),z;t)\le 1-\varepsilon$ and $\mathsf{F}(t_{m,n}(x)-L,z;\frac{t}{2})>1-q,$ in particular $\mathsf{F}(t_{f,g}(x)-L,z;\frac{t}{2})>1-q.$ Then

$$\begin{aligned}1-\varepsilon&\ge\mathsf{F}(t_{m,n}(x)-t_{f,g}(x),z;t)\\&\ge\tau\left(\mathsf{F}\left(t_{m,n}(x)-L,z;\frac{t}{2}\right),\mathsf{F}\left(t_{f,g}(x)-L,z;\frac{t}{2}\right)\right)\\&>\tau((1-q),(1-q))>1-\varepsilon,\end{aligned}$$

which is imposible. Therefore $B_\varepsilon(\lambda,\mu)\subset A_q(\lambda,\mu)$. Hence, by (3) $\delta_2(B_\varepsilon(\lambda,\mu))=0.$ Therefore, x is statistically (λ,μ)-Cauchy in PTN-space.

Definition 8. *Let* (X,F,τ) *be a PTN space. Then,*

(a) *PTN-space is said to be complete if every Cauchy double sequence is* P*-convergent in* (X,F,τ).

(b) *PTN-space is said to be statistically* (λ,μ)*-complete (or briefly,* $\mathsf{F}(st_{\lambda,\mu})$*-complete) if every statistically* (λ,μ)*-Cauchy sequence in PTN space is statistically* (λ,μ)*-summable.*

Theorem 5. *Every probabilistic* 2 *-normed space* $(X, \mathbf{F}, \tau)$ *is* $\mathbf{F}(st_{\lambda,\mu})$*-complete but not complete in general.*

Proof. Assume that $x=(x_{jk})$ is $\mathbf{F}(st_{\lambda,\mu})$-Cauchy but not $\mathbf{F}(st_{\lambda,\mu})$-summable. Then there exists $M, N \in \mathbb{N}$ such that for all $m, p \geq M$, $n, q \geq M$, the set

$$D_{\varepsilon}(\lambda,\mu)=\{(m,n)\in \mathbb{N}\times\mathbb{N} \,:\, \mathbf{F}(t_{m,n}(x)-t_{p,q}(x), z; t)\leq 1-\varepsilon\}=0$$

has double natural density zero, i.e. $\delta_2(E_{\varepsilon}(\lambda,\mu))=0$ and

$$\delta_2(E_{\varepsilon}(\lambda,\mu))=\delta_2\left(\left\{(m,n)\in \mathbb{N}\times\mathbb{N} \,:\, \mathbf{F}\left(t_{m,n}(x)-L, z; \frac{t}{2}\right)>1-\varepsilon\right\}\right)=0.$$

It follows that $\delta_2(E^c_{\varepsilon}(\lambda,\mu))=1.$ Since

$$\mathbf{F}(t_{m,n}(x)-t_{p,q}(x), z; t)\geq 2\mathbf{F}\left(t_{m,n}(x)-L, z; \frac{t}{2}\right)>1-\varepsilon,$$

if $\mathbf{F}(t_{m,n}(x)-L, z; \frac{t}{2})>\frac{1-\varepsilon}{2}$. Hence $\delta_2(E^c_{\varepsilon}(\lambda,\mu))=0,$ which give rise to a contradiction, since $x=(x_{jk})$ is $\mathbf{F}(st_{\lambda,\mu})$-Cauchy. Consequently, $x=(x_{jk})$ must be $\mathbf{F}(st_{\lambda,\mu})$-summable.

To see that a probabilistic 2 -normed space is not complete in general, for this, we have the following example:

Example 2. $X=(0,1]\times(0,1]$ *and* $\mathbf{F}(x,z;t)=\frac{t}{t+\|x,z\|}$ *for* $t>0$ *and nonzero* $z\in X$. *Then* $(X, \mathbf{F}, \tau)$ *is a probabilistic* 2 *-normed space but not complete, since the double sequence* $\left(\frac{1}{mn}\right)$ *is Cauchy with respect to* $(X, \mathbf{F}, \tau)$ *but not* P *-convergent with respect to the present PTN-space.*

IV. Conclusion

This study indeed presents a relationship between two various disciplines: the theory of probabilistic normed spaces and summability theory. Some new type of summability methods for double sequences involving the ideas of de la Vallée-Poussin mean has not been studied previously in the setting of probabilistic 2 -normed (PTN) spaces. Motivated by this fact, in this paper, the notion of (λ,μ)-summable, statistically (λ,μ)-summable, statistically (λ,μ)-Cauchy and statistically (λ,μ)-complete for double sequence with respect to PTN-space and establish some interesting results. These results can be utilized to study the convergence problems of double sequences having chaotic pattern in probabilistic 2 -normed spaces.

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