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Runoff as a Stochastic Process

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Abstract—**Runoff stationary critical flow is investigated as a stochastic process by means of two routing simulation models, a stream confluence, which has beens interpreted as a Marcus-Lushnikov coalescence process, and a channel splitting model, which has ben interpreted as a Markov chain over a regular tree. Despite of the expected similarity due to expection that they should be seen as one the backward of the other, the initiation and the stopping methods using in algorithms influence strongly stream size distribution.**

Index Terms—**runoff, stochastic processes**

I. Introduction

Runoff is a fundamental process in surface hydrology, related to phenomena as erosion, landslides and flooding, all issues of growing importance for civil risks and food security in a climate-change scenario.

Runoff takes place at any rainfall event and its intensity and duration are related to scale and period of runoff: in practice it occurs when rainfall rate is greater than infiltration rate, and when the process reach a certain scale it is the main driver of hydrological network, both feeding and forming it. At basin scale routing models are widely used to forecast how streams collect water from its catchment area: in such approaches the watershed is represented in terms of homogeneous surfaces where runoff is interpreted by simple dynamics as the mean-field Kinematic Wave one ([4],[1]).

Though the complexity of routing models justify such lumped approach, physically-based models have been continuously chased as an El-Dorado ([6]) because of the need to describe the very nature of runoff process mat a field scale.

One of major problem in developing a physical model of runoff stay in surface complexity, which is usually represented in terms of roughness, a concept related to a number of measurable variables (e.g. pocket density, average depth, ..) with a strong spatial variability.

The present approach is aimed to identify which stochastic processes are suitable to model runoff at such a scale, an issue which is faced interfacing the phenomenological description to stochastic language by means of simulation models.

Phenomenology

Runoff is a process occurring over a surface which normally owns depressions (pockets) with a size ranging from the scale of mm to that of dm.

During a rainfall randomly falling droplets are collected by those depressions, and when rainfall intensity exceeds pocket leakage (infiltration) a cascade begins 1.

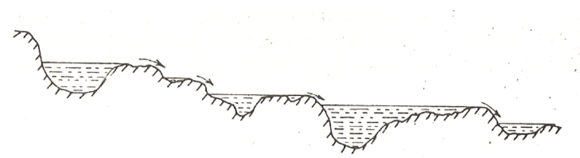

Fig. 1. Sketch of pocket cascade (from [1])

As those depression have different height, capacity and location, spilling occurs in a random direction feeding the closest pocket. Streams merges and increase in size and sone generate rills and channels 2..

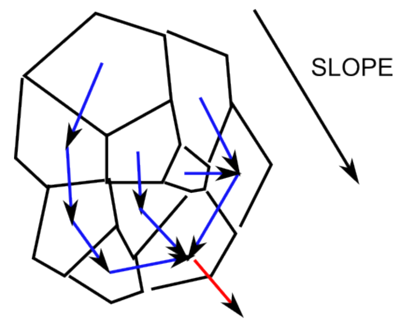

Fig. 2. Divide of a surface forming a pocket cascade

In 2D view runoff field may be seen as made of pockets whose centers are located randomly as nodes of a lattice.

Toy model #1

To simulate runoff a depression lattice can be easily generated by means of a random (uniform) distribution, while Thiessen - Voronoi geometrical approach can be used to generate a polygonal divide of the surface (see figure 3). From the catchment area boundary poligons are conveniently removed.

The generated coverage is related to a pocket size distribution reported in figure 4 (right side).

Such a coverage is used to produce a discharge network, assigning each node (pocket) a downstream child on the base of slope (giving the main flow direction), distance and angle.

Stream network generation procedure is based on two steps:

- for each node (pocket) identification of downward child node is made whose multiplicity is increased.
- starting from nodes with multiplicity 0 (source pockets), streams are drawn to child nodes whose resolved multiplicity is reduced while increasing the size of the stream that will emerge from it;
- the last step is repeated for the nodes reaching a residual multiplicity equal to 0, till zeroing the nodes.

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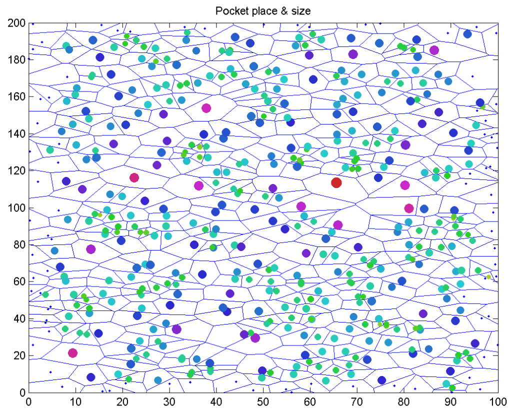

Fig. 3. 2D distribution of 500 pockets generated in Matlab with ***random('uniform')*** function and their Thiessen-Voronoi spatialization

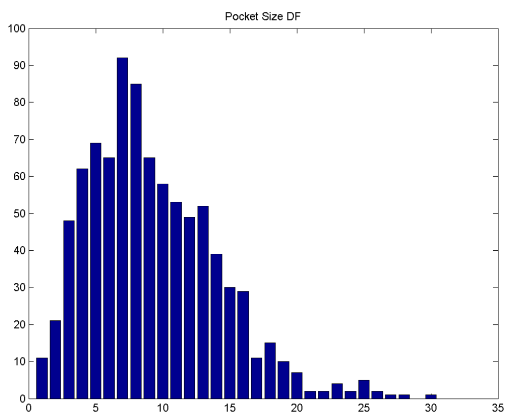

Fig. 4. Distribution of pocket size

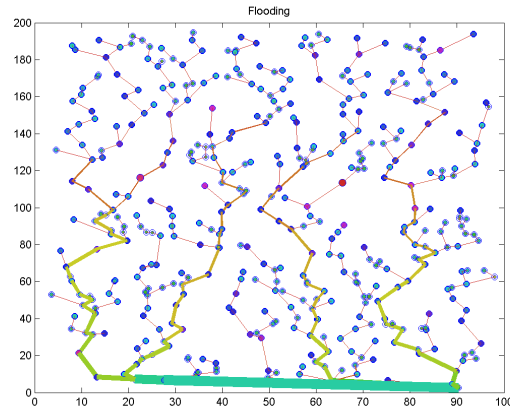

Fig. 5. Criptical flow routing obtained from the divide in figure 3

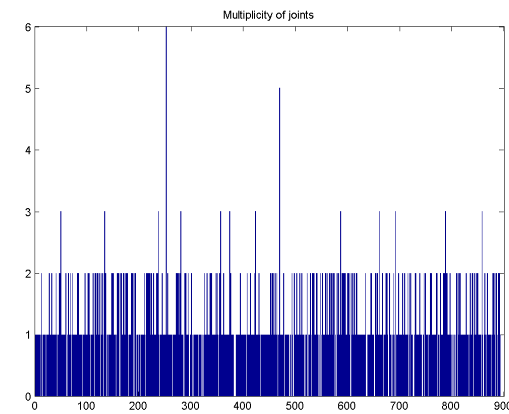

Fig. 6. Confluence multiplicity of nodes in figure 3

Such an algorithm does not simulate true dynamics, as confluence nodes stay in a dormant state till every uphill nodes are waken up (residual multiplicity goes to 0).

So doing a routing network is generated corresponding to **stationary critical flow** that is a full stream saturation under a constant rainfall regime without considering surface floods (stream merging) and erosion **(hard surface** assumption). Under these hypothesys it is possible to identify two kind of pockets, 'sources' (tree leaves) which feed the network, and 'confluence nodes' representing those pockets where streams merge to one another 5.

Node multiplicity is shown in figure 6 where, apart a large number of 'sources'a certain amount of nodes with more than 2 confluences are also present.

Figure 7 reports the distributions of stream size (flow).

Toy Model #2

Another way of generating a runoff routing can be easily obtained as a binary branching scheme going back from the outlet channel. Figure 8 has been obtained by successive random bifurcations (split has been obtained using a uniform distribution), stopping the process when the branch size reaches 1/1000 of that of main stream; branches total to around 4000.

Model #2 as much as Model #1 can be useful to develop rainfall-runoff models at field scale to study the response to roughness.

Both models support a delay which can be easily related to stream size, though it doesn't perform any leaf node spatializatiom: to assign each graph leaf a pocket without leaving empty spaces, additional rules should be added further (e.g. as in basin-scale models, [5]).

II. Coalescence and Branching processes

The models used above make runoff similar to other phenomena widely studied in the literature in terms of stochastic processes, as coalescence of particles in dispersed media (aerosols and hydrosols), formation of agglomeration of emulsion droplets (oil separation) or coagulation (before gelation), solid state avalanche breakdown in electronics.

Marcus-Lushnikov model

Model#1 can be seen as a Marcus-Lushnikov (ML) coalescence process, where a couple of particles (streams) with a

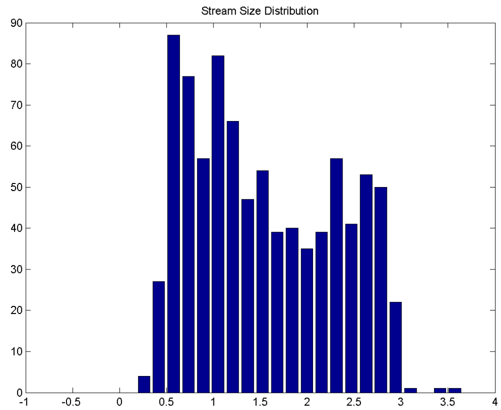

Fig. 7. Distribution of log of stream size

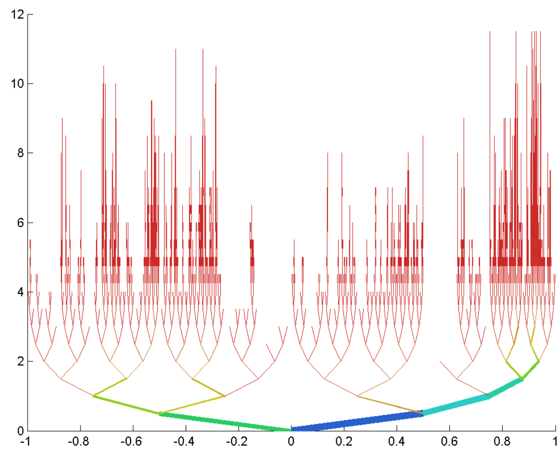

Fig. 8. Bottom-up reconstruction of a runoff process; x-axis is an arbitrary cohordinate systems used just for display purposes whereas on y-axis is the number of splits

given mass (flow) coalesce (merge) to a new entity conserving their mass:

$$(x) + (y) \rightarrow (x + y)$$

It is a particular Markov chain process considering $n(> 1)$ finite mass particles of mass $(x_1, .., x_n) \in (0, \infty)^n$, with total mass $M(= \sum_{i=1,n} x_i)$.

Such a process has been formerly described from Lushnikov ([2]) introducing the state of the system:

$$Q = n_1, .., n_g, ..$$

which is a mass distribution function where n_g is the number of particles of size g : each time a coalescence event takes place, two bins decrease of one unit and one of them increases of one unit

$$Q^+ = n_1, .., n_l - 1, ..n_m - 1, ..n_{l+m} + 1, ..$$

therefore the mass distribution integral $|Q| = \sum n_i$ is not time-conservative, while the total mass $M = \sum y_i \cdot n_i$does.

In the case of runoff, assuming that particles (source pockets/streams) of the same size, namely $n(x, 0) = N \cdot \delta(x_0)$, the process is made a discrete one, and all the downhill stream sizes are multiple (g is multiplicity) of the source one (which can be easily related to roughness).

Therefore the distribution transformation process: $Q \rightarrow Q^+$ can be represented by the **rate** $A(Q^+, Q)$:

$$A(Q^+, Q) = K_{l,m} \cdot n_l(Q) \cdot [n_m(Q) - \delta(l, m)]$$

where $n_g(Q)$ is the number of particles with size g in the state Q, and $K_{l,m}$ is the probability of coalescence of two streams of size l and m respectively.

As model #1 is describing a **steady state regime,** Marcus-Lushnikov process should be considered as a space process; however as it is not possible to observe every particles (streams) of the system simultaneously, we have to look at the process in terms of strips of length δz where an injection of 'new' source streams occur and the total number of streams is conserved:

$$Q \quad \rightarrow \quad Q^+ + \{N - |Q^+|, 0, .., 0\}$$

where$|Q|$ is the number of streams in the considered state.

Splitting model

Even if split process (also know as binary branching/Galton-Watson p.) seems the better candidate to represent model #2, it is also obvious that branching is not an option as split probability refers to the partitioning of a particle of mass l:

$$(l) \rightarrow (x) + (l - x) \quad ; \quad 0 < x < l$$

Therefore partitioning occurs on a graph which in first instance can be assumed to be a regular tree, and the process is a Markov chain on a tree (see e.g. [3]) where, at each branching, a new Markov chain is initiated. Therefore in a backward runoff process it applies repetitively, starting from the main channel, which after n steps reduces its size by:

$$r_n = x_1 \cdot \ldots \cdot x_i \cdot y_1 \cdot \ldots \cdot y_{n-i} \quad : \quad y = 1 - x$$

In fact runoff Model #2 has some discrepancies with such a split process:

- stream length is not taken into account so probability of branching after a fixed roughness dependent time/space interval is considered to be 1;
- at branching the partitioning probability density is random (uniform);
- branch production is stopped as a roughness-dependent size is reached, that is when $L \cdot r_n < L_0$.

As a dead event takes place, the particle is no more split, therefore the process stops. Distribution of stream size is shown in the figure 9 where after applying a log transformation to size, a gaussian-like behavior appear, which suggests an interpretation in terms of diffusive models.

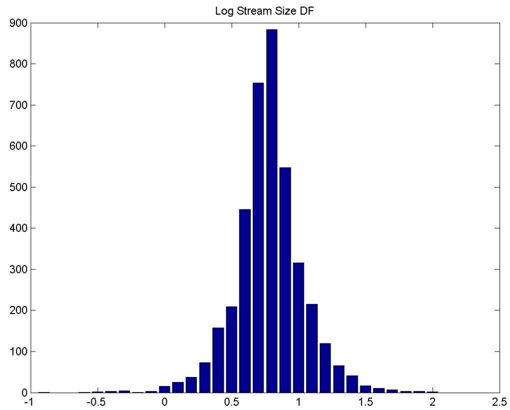

Fig. 9.

III. Conclusions

The study analyzes the possibility to represent runoff in terms of a couple of stochastic processes, the coalescence (Marcus-Lushnikov) and the splitting one, revealing that the former one could be used once introducing an injection of particles to conserve their number. The process which can be used to represent the inverse runoff model could be a simple Markov chain over a regular tree, with a death term to consider the reaching of a stream of its birth point. Even if both routing models seem to be sound with the physical interpretation, initiation and stopping methods used in routing algorithms don't allow them to produce the same distributions of stream sizeo.

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