

Some advantages of the hybrid methods, which used the first derivative of the solution of the considered problem

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Abstract— The scientists began investigate of the solution of ODE from the XVII century. Many famous mathematicians as Newton, Leibniz, Bernoulli, D' Alembert, Euler, Cauchy and etc. have considered solving of the ordinary differential equations. For solving these equations the scientists from the different country have constructed many methods. Here, with the help of the compare known results are shown some advantages of the hybrid methods. Are constructed the hybrid methods with the order of accuracy $p = 8k$. Here, is suggested concrete hybrid method of the fractional step type with the order of accuracy $p = 8$ for $k = 1$ for which uses the first order derivative of the solution of the initial value problem.

Keywords—Initial value problem for ODE, hybrid methods, degree and stability of the hybrid methods, the relation between of the degree and order of the hybrid methods.

I. INTRODUCTION

There is a wide arsenal of numerical methods for solving ordinary differential equation of the first order. Remark, that like Clairaut, many scientists have applied of the indirect-numerical method investigates practical problems (see, for example [1, p.132-133]). However, Euler found the shortcomings of existing methods and has constructed a direct method, which now is called the explicit Euler method (see [2, p. 289]). Euler has determined also the shortcomings of his method and suggested two ways to correct the indicated shortcomings. One of them is the use of Taylor's formula. The

replacing the higher-order derivatives in Taylor's formula the first derivative, Runge-Kutta (in the beginning of XX century) and Adams (in the middle of XIX century) have constructed numerical methods, that generalizes the one and multistep methods accordingly. However, to solving of the following initial-value problem:

$$y' = f(x, y), \quad y(x_0) = y_0. \quad (1)$$

Here proposed use of the second derivative of the solution of problem (1) for construction of hybrid methods.

It is known that the Runge-Kutta methods, which applied to solving of problem (1), may be written as follows:

$$y_{n+1} = y_n + h \sum_{i=1}^s b_i K_i^{(s)}, \quad (2)$$

$$K_i^{(s)} = f(x_n + \alpha_i h, y_n + h(\beta_{1,s} K_1^{(s)} + \beta_{2,s} K_2^{(s)} + \dots + \beta_{s,s} K_s^{(s)})), \\ i = 1, 2, \dots, s.$$

Usually this method is called the implicit Runge-Kutta method.

Let us consider the generalization of the Adams method, which can be written as the following:

$$\sum_{i=0}^k \alpha_i y_{n+i} = h \sum_{i=0}^k \beta_i f_{n+i}. \quad (3)$$

In references, this method is called the k -step method with constant coefficients. Note that, there are some relationships between methods Runge-Kutta and multistep (see for example [3], [4]). But the implicit Runge-Kutta methods correspond to the forward-jumping methods, which are received from the method (3) for the values $\alpha_k = \alpha_{k-1} = \dots = \alpha_{k-m+1} = 0$ and $\alpha_{k-m} \neq 0$. As follows from this for the application of forward-jumping methods may be replaced by application of implicit Runge-Kutta methods. Therefore taking into account of the some advantages of forward-jumping methods in the work [5] considered of application of these methods to solving Volterra integral equations.

In the middle of the XX century, scientists have constructed procedures which are called hybrid methods. One of papers devoted to the construction of hybrid methods

belongs to Gear (see [6]-[10]). In general form this method may be written as follows:

$$\sum_{i=0}^k a_i y_{n+i} = h \sum_{i=0}^k \beta_i f_{n+i} + h \beta f_{n+k-\nu} \quad (0 < \nu < 1). \quad (4)$$

Gear's method is a hybrid method of type (3). One of the first hybrid methods of type (2) was constructed in [11] and has the following form:

$$y_{n+1} = y_n + h(K_1 + K_2)/2. \quad (5)$$

where

$$K_1 = f(x_n + \frac{3-\sqrt{3}}{6}h, y_n + h(K_1 + \frac{3-2\sqrt{3}}{3}K_2)/4);$$

$$K_2 = f(x_n + \frac{3+\sqrt{3}}{6}h, y_n + h(\frac{3+2\sqrt{3}}{3}K_1 + K_2)/4).$$

Remark that the method (5) constructed on the basis of trapezoidal method and belongs to a class of implicit Runge-Kutta methods. For using the methods (5) here have proposed the following scheme, which recalls a predictor-corrector methods applying to investigation of the implicit multistep methods:

$$K_1 = f(x_n + \frac{3-\sqrt{3}}{6}h, y_n + h(\hat{K}_1 + \frac{3-2\sqrt{3}}{3}\hat{K}_2)/4);$$

$$K_2 = f(x_n + \frac{3+\sqrt{3}}{6}h, y_n + h(\frac{3+2\sqrt{3}}{3}\hat{K}_1 + \hat{K}_2)/4).$$

Obviously, if $\hat{K}_1 = K_1$ and $\hat{K}_2 = K_2$ from this we receive the method (5). But if the quantity \hat{K}_1 and \hat{K}_2 is defined as:

$$\hat{K}_1 = f(x_n + \frac{3-\sqrt{3}}{6}h, y_n + \frac{3-\sqrt{3}}{6}hf(x_n + \frac{3-\sqrt{3}}{12}h, y_n + \frac{3-\sqrt{3}}{12}f_n)),$$

$$\hat{K}_2 = f(x_n + \frac{3+\sqrt{3}}{6}h, y_n + \frac{3+\sqrt{3}}{6}hf(x_n + \frac{3+\sqrt{3}}{12}h, y_n + \frac{3+\sqrt{3}}{12}f_n)),$$

then one can be obtain the explicit method. However, for a simpler method the quantity \hat{K}_1 and \hat{K}_2 may be determined as follows:

$$\hat{K}_1 = f(x_n + (3-\sqrt{3})h/6, y_n + (3-\sqrt{3})hf_n/6),$$

$$\hat{K}_2 = f(x_n + (3+\sqrt{3})h/6, y_n + (3+\sqrt{3})hf_n/6).$$

Now consider the determination of the quantity K_1 and K_2 in the following form:

$$K_1 = f(x_{n+1/2-\alpha}, y_{n+1/2-\alpha}),$$

$$K_2 = f(x_{n+1/2+\alpha}, y_{n+1/2+\alpha}) \quad (\alpha = \sqrt{3}/6).$$

In this case the method (5) is converted to a symmetric hybrid method of the multistep type.

Note that often the hybrid methods are constructed in the symmetric form. Similar methods may be written in the following general form (see for example [7]-[10]):

$$\sum_{i=0}^k \alpha_i y_{n+i} = h \sum_{i=0}^k \beta_i f_{n+i} + h \hat{\beta}_0 f_{n+m+s} + h \hat{\beta}_k f_{n+k+m-s},$$

($s > 0; m-s > 0; m+s \leq k$).

It is clear that hybrid method in simple form can be written as:

$$\sum_{i=0}^k \alpha_i y_{n+i} = h \sum_{i=0}^k \beta_i f_{n+i+l_i} \quad (|l_i| < 1). \quad (6)$$

Now consider to construction of the methods on junction of the methods (6) and the methods of Adams.

II. ON ONE GENERALIZATION OF THE METHOD (6)

This method is more precise than corresponding classic methods of Runge-Kutta and Adams. If to generalize this, then in result we receive the following:

$$\sum_{i=0}^k \alpha_i y_{n+i} = h \sum_{i=0}^k \beta_i f_{n+i} + h \sum_{i=0}^k \gamma_i f_{n+i+l_i} \quad (|l_i| < 1). \quad (7)$$

Obviously, that the method (7) - is the hybrid method of the multistep type with the constant coefficients. Note that the integer-valued quantity p is called the degree of the method (7) if the following asymptotic equality is holds:

$$\sum_{i=0}^k (\alpha_i y(x+ih) - h \sum_{i=0}^k (\beta_i y'(x+ih) + \gamma_i y'(x+(i+l_i)h))) = O(h^{p+1}), \quad h \rightarrow 0. \quad (8)$$

Note that usually the definition of the order of accuracy of the hybrid methods of multistep type is identical of the definition of the degree of ordinary multistep methods.

The stability of method (7) may be defined according to the definition of the stability for multistep methods (see [12]). One of the basic issues in the investigation of the this method is the relationship between its degree and order. Before determining the relationship, let us consider some restrictions imposed on the coefficients of method (7).

A: The coefficients $\alpha_i, \beta_i, \gamma_i$ and $l_i, (i=0,1,2,\dots,k)$ are the real numbers; moreover, $\alpha_k \neq 0$.

B: The characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^k \alpha_i \lambda^i; \quad \sigma(\lambda) \equiv \sum_{i=0}^k \beta_i \lambda^i; \quad \gamma(\lambda) \equiv \sum_{i=0}^k \gamma_i \lambda^{i+l_i}$$

have no common multiple different from constant.

C: $\sigma(1) + \gamma(1) \neq 0$ and $p \geq 1$.

With the help of the methods of undetermined coefficients, we can examine the definition of the quantities $\alpha_i, \beta_i, \gamma_i, l_i$ ($i=0,1,2,\dots,k$), so we will consider the following expansion:

$$y(x+ih) = y(x) + ihy'(x) + \frac{(ih)^2}{2!} y''(x) + \dots + \frac{(ih)^p}{p!} y^{(p)}(x) + O(h^{p+1}), \quad (9)$$

$$y'(x+ih) = y'(x) + ihy''(x) + \frac{(ih)^2}{2!}y'''(x) + \dots + \frac{(ih)^{p-1}}{(p-1)!}y^{(p)}(x) + O(h^p), \quad (10)$$

where $x = x_0 + nh$ is a fixed point.

Note that the values of the coefficients of the multistep method in some sense are connected with the relationship between the order and degree of method (7); therefore, we require the following lemma.

Lemma. Let $y(x)$ be a sufficiently smooth function, and assume that conditions A, B, and C are holds. In order to make the method (7) had a degree p , satisfaction of its coefficients of the following conditions are necessary and sufficient:

$$\sum_{i=0}^k \alpha_i = 0, \quad \sum_{i=0}^k i\alpha_i = \sum_{i=0}^k (\beta_i + \gamma_i),$$

$$\sum_{i=0}^k \frac{i^l}{l!} \alpha_i = \sum_{i=0}^k \frac{i^{l-1}}{(l-1)!} \beta_i + \sum_{i=0}^k \frac{(i+l_i)^{l-1}}{(l-1)!} \gamma_i, \quad (11)$$

$$(l = 2, 3, \dots, p).$$

Proof. We first prove that if method (7) has the degree p , then the coefficients $\alpha_i, \beta_i, \gamma_i, l_i$ ($i = 0, 1, 2, \dots, k$) will satisfy the system of nonlinear algebraic equations given in (11).

Taking into account that the method (7) has the degree p , then by using Taylor expression (9) and (10) in the left hand-side of asymptotic equality (8). We have:

$$\left(\sum_{i=0}^k \alpha_i \right) y(x) + h \sum_{i=0}^k (i\alpha_i - \beta_i - \gamma_i) y'(x) +$$

$$+ h^2 \sum_{i=0}^k \left(\frac{i^2}{2} \alpha_i - i\beta_i - (i+l_i)\gamma_i \right) y''(x) + \dots +$$

$$+ h^p \sum_{i=0}^k \left(\frac{i^p}{p!} \alpha_i - \frac{i^{p-1}}{(p-1)!} \beta_i - \frac{(i+l_i)^{p-1}}{(p-1)!} \gamma_i \right) y^{(p)}(x) = O(h^{p+1}),$$

$$h \rightarrow 0.$$

If take in account that the method (7) has the degree p , then we obtain the following:

$$\sum_{i=0}^k \alpha_i y(x) + h \sum_{i=0}^k (i\alpha_i - \beta_i - \gamma_i) y'(x) +$$

$$+ h^2 \sum_{i=0}^k \left(\frac{i^2}{2} \alpha_i - i\beta_i - (i+l_i)\gamma_i \right) y''(x) + \dots$$

$$+ h^p \sum_{i=0}^k \left(\frac{i^p}{p!} \alpha_i - \frac{i^{p-1}}{(p-1)!} \beta_i - \frac{(i+l_i)^{p-1}}{(p-1)!} \gamma_i \right) y^{(p)}(x) = 0. \quad (13)$$

It is known that $1, x, x^2, \dots, x^p$ forms a linearly independent system; therefore, equality (13) is equivalent to the following:

$$\sum_{i=0}^k \alpha_i = 0, \quad \sum_{i=0}^k (i\alpha_i - \beta_i - \gamma_i) = 0, \dots, \quad (14)$$

$$\sum_{i=0}^k \left(\frac{i^p}{p!} \alpha_i - \frac{i^{p-1}}{(p-1)!} \beta_i - \frac{(i+l_i)^{p-1}}{(p-1)!} \gamma_i \right) = 0.$$

We now will prove that if the coefficients of the method (7) are the solution of the nonlinear system (11), then its degree is equal to p . Indeed, if we used the system of equalities of (14) into equality (12), then we obtain the asymptotic equality of (8). From this asymptotic equality it follows that method (7) must have the degree p . It is easy to determine that for the chosen values $l_i = 0$ ($i = 0, 1, \dots, k$), the system (11) is linear and coincides with the known systems used for defining the coefficients of the multistep method with constant coefficients. Subject to the conditions from $|l_0| + |l_1| + \dots + |l_k| \neq 0$, the system (11) is nonlinear. This system contains $p+1$ equations in $4k+4$ unknowns and is homogeneous; it must possess the zero solution, and for system (11) have a non-zero solution, suppose that the condition $4k+4 > p+1$ is holds. Hence, we obtain that there are methods of type (7) with the order $p \leq 4k+2$.

Consider the construction methods of type (6) and suppose that $k=1$. Then, under the assumption that $\alpha_1 = -\alpha_0 = 1$ and $\gamma_1 = \gamma_0 = 0$, from the system of (11) we will have the following system for the determined of variables β_0, β_1, l_0 and l_1 :

$$\beta_0 + \beta_1 = 1,$$

$$l\beta_0 + \gamma\beta_1 = 1/2,$$

$$l^2\beta_0 + \gamma^2\beta_1 = 1/3,$$

$$l^3\beta_0 + \gamma^3\beta_1 = 1/4. \quad (15)$$

Here, $l = l_0$, and $\gamma = 1 + l_1$. Solving this nonlinear system of equations for l and γ in result receive the following quadratic equation:

$$l^2 - l + 1/6 = 0.$$

The value of γ is determined from the equation $\gamma + l = 1$. Note that the method with the degree $p = 4$ can be written as follows:

$$y_{n+1} = y_n + h(f_{n+l_0} + f_{n+1+l_1})/2. \quad (16)$$

Here $l_1 = -l_0$; $l_0 = (3 - \sqrt{3})/6$, $1 + l_1 = (3 + \sqrt{3})/6$. For this aim consider to applying hybrid method (16). Need to know the value of the $y_{n+1/2+\sqrt{3}/6}$, y_{n+l_0} and $y_{n+1/2-\sqrt{3}/6}$.

Note, that these variables are independent of y_{n+1} , because that method (16) is explicit. But therefore, there still exist implicit hybrid methods. For example, consider the following method:

$$y_{n+1} = y_n + h(3f_{n+1/3} + f_{n+1})/4. \quad (17)$$

This method is an implicit hybrid method with degree $p = 3$ and is A-stable (see [13]).

Consider method (7) for $k=1$. In this case, assuming that $\alpha_1 = -\alpha_0 = 1$, the system (15) takes the following form:

$$\begin{aligned}\beta_0 + \beta_1 + \gamma_0 + \gamma_1 &= 1, \\ \beta_1 + l_0\gamma_0 + l_1\gamma_1 &= 1/2, \\ \beta_1 + l_0^2\gamma_0 + l_1^2\gamma_1 &= 1/3, \\ \beta_1 + l_0^3\gamma_0 + l_1^3\gamma_1 &= 1/4, \\ \beta_1 + l_0^4\gamma_0 + l_1^4\gamma_1 &= 1/5, \\ \beta_1 + l_0^5\gamma_0 + l_1^5\gamma_1 &= 1/6.\end{aligned}$$

The solution of this nonlinear system yields the following:

$$\begin{aligned}\beta_0 = \beta_1 &= 1/12, \quad \gamma_0 = \gamma_1 = 5/12, \\ l_0 &= 1/2 - \sqrt{5}/10, \quad l_1 = 1/2 + \sqrt{5}/10.\end{aligned}$$

The corresponding method with degree $p=6$ takes the following form:

$$y_{n+1} = y_n + h(f_{n+1} + f_n)/12 + 5h(f_{n+1/2-\sqrt{5}/10} + f_{n+1/2+\sqrt{5}/10})/12 \quad (p=6) \quad (18)$$

To apply hybrid methods to solving of some problems, we should know the values of $y_{n+1/2-\sqrt{5}/10}$ and $y_{n+1/2+\sqrt{5}/10}$, and the accuracy of these values should have order at least $O(h^6)$. Note that hybrid method (18) is implicit and that when applying it to solving of initial value problem (1), is used a predictor-corrector scheme containing only one explicit method. Therefore, we consider the construction of an explicit method (in one variant) has the following form:

$$\begin{aligned}y_{n+1} &= y_n + hf_n/9 + h((16 + \sqrt{6})f_{n+(6-\sqrt{6})/10} + \\ &+ (16 - \sqrt{6})f_{n+(6+\sqrt{6})/10})/36.\end{aligned} \quad (19)$$

This method is explicit and has degree $p=5$.

To use method (19) we must define $y_{n+(6-\sqrt{6})/10}$ and $y_{n+(6+\sqrt{6})/10}$. The technique used to calculate these quantities determines the properties of the block method.

Suppose that the approximated values of the solution of problem (1) the mesh points $x_n + (1 - \sqrt{5}/5)h/2$ and $x_n + (1 + \sqrt{5}/5)h/2$ have been identified by some method. Then (18) may be considered as equation in the unknowns y_{n+1} , whose solution is usually obtained via iterative processes. In contrast, we suggest a predictor-corrector method, which recalls block methods. It is easy to show that first one can calculate the values of according to method (19) and then correct these values by the method (18). We therefore constructed an algorithm for applying method (19) to solving of problem (1).

Note that can be acquainted with application of hybrid methods to numerical solution of Volterra integral and integro-differential equations in [14]-[16]. And in [17] for solving problem (1) is constructed hybrid method with degree $p=7$ for $k=3$ by using collocation approach.

It follows that the investigation of hybrid methods is more difficult than the known. Consider the following hybrid method:

$$\begin{aligned}y_{n+1} &= y_n + h(64y'_{n+1} + 98y'_{n+1/2} + 18y'_n)/360 + \\ &+ h(18y'_{n+1/2+\beta/2} + 98y'_{n+1/2} + 64y'_{n+1/2-\beta/2})/360, \quad (20) \\ &(\beta = \sqrt{21}/14).\end{aligned}$$

This stable method is one step and have the degree $p=8$. In the next we show that this method is not received from the method (7) as the special case.

Remark that, the method (20) is available for solving problem (1), which is not contained in the method (7) as the special case. But if we change h by $2h$ and use it in the formula (20), then we receive the method, which is including in class methods of type (7). Note that this method can be written as the following:

$$\begin{aligned}y_{n+2} &= y_n + h(64y'_{n+2} + 98y'_{n+1} + 18y'_n)/180 + \\ &+ h(18y'_{n+1+\beta} + 98y'_{n+1} + 64y'_{n+1-\beta})/180.\end{aligned} \quad (21)$$

Now consider the construction of some procedure for solving problem (1) by using hybrid methods with the degree $p=4$, $p=5$ and $p=6$. Note that these methods are constructed for $k=1$ (the characteristic polynomial has one root which is define as the $\lambda=1$), and all of them are stable. Methods (16) and (19) are explicit, whereas method (18) is implicit. However, the application of explicit hybrid methods requires some additional auxiliary formulas. To this end, we construct an algorithm for method (16).

Algorithm 1. Applies method (16) to the solution of problem (1).

Step 1. Calculate y_{n+l} and y_{n+1-l} by with the following block method:

$$\begin{aligned}y_{n+l} &= y_n + lhf_n, \\ \hat{y}_{n+l} &= y_n + lh(f_n + \bar{f}_{n+l})/2, \quad (\bar{f}_m = f(x_m, \bar{y}_m), \quad m=0,1,2,\dots), \\ y_{n+l} &= y_n + lh(5f_n + 8\hat{f}_{n+l})/12 - \\ &- lhf(x_n + 2lh, y_n + 2lh\hat{f}_{n+l})/12, \\ (\hat{f}_m &= f(x_m, \hat{y}_m), \quad m=0,1,2,\dots).\end{aligned}$$

Repeat this scheme for $l := (3 - \sqrt{3})/6$ and $1+l = (3 + \sqrt{3})/6$

Step 2. Calculate y_{n+1} according to method (16). Here, we compute the values of the quantities y_{n+l} and y_{n+1-l} to within $O(h^4)$, which suffices for this algorithm.

Now, we construct an algorithm for applying method (19).

Algorithm 2. Applies method (19) to the numerical solution of problem (1), assuming that the values y_0 and $y_{1/2}$ have been determined with the required accuracy.

Step I. Set $\hat{y}_{n+1} = y_n + hy'_{n+1/2}$,

Step II. Set $y_{n+1} = y_n + h(\hat{y}'_{n+1} + 4y'_{n+1/2} + y'_n)/6$,

Step III. Set $y_{n+3/2} = y_{n+1/2} + h(7y'_{n+1} - 2y'_{n+1/2} + y'_n)/6$,

Step IV. Compute

$$y_{n+\alpha} = y_n + \alpha h y'_n + \alpha^2 h ((\alpha^2 - 12\alpha + 6)y'_{n+3/2} - (3\alpha^2 - 48\alpha + 27)y'_{n+1} + (3\alpha^2 - 60\alpha + 54)y'_{n+1/2} - (\alpha^2 - 24\alpha + 33)y'_n) / 18$$

$$\text{for } \alpha = \frac{6 - \sqrt{6}}{10} \text{ and } \alpha = \frac{6 + \sqrt{6}}{10}.$$

Step V. Conclude that

$$y_{n+1} = y_n + h f_n / 9 + h((16 + \sqrt{6})f_{n+(6-\sqrt{6})/10} + (16 - \sqrt{6})f_{n+(6+\sqrt{6})/10}) / 36.$$

For demonstrate algorithm 1 consider to application of its to the solving of the next problem:

$$y' = \cos x, \quad y(0) = 0 \quad (\text{Exact solution as } y(x) = \sin x).$$

Results tabulated in table 1.

Step size	Variable x	Error of the algorithm 1
$h = 0.05$	0.10	$0.14E - 09$
	0.40	$0.56E - 09$
	0.70	$0.93E - 09$
	1.00	$0.12E - 08$

Conclusion. We have constructed a multistep hybrid method with the degree $4 \leq p \leq 6$ for $k = 1$, and with the degree $p = 8$ for $k = 2$. It is known that for $k = 1$, the ordinary k-step method has maximal degree $p_{\max} = 2$, which yields a trapezoidal method. However, the hybrid approach, which constructed here has maximal degree $p_{\max} = 6$, although the application of the trapezoid method is simpler than applying hybrid procedure. Using the Euler explicit method in place of the predictor method, one can construct a predictor-corrector scheme for the practical application of the trapezoidal method. Remark that for the construction of the stable methods with the degree $p = 2k + 2$ can be used multistep methods with second derivatives (see, for example [15]-[25]). In this paper, we have used the block method to construction of algorithms for using hybrid methods. The method, having the degree $p = 4$, was described in algorithm 1, and algorithm 2 formulated to using of the method (19). Note that after some modifications, algorithm 2 may be realized for using method (18). In the end, let us remark that for $k = 2$, we have constructed stable methods of type (7) with degree $p = 8$ and by using that shown how can be receive the one step method (20) from the two step method (21). Method (20) is one step hybrid method and different from method (7). Consequently the methods of type (20) are more accurate, than the methods of type (7). Thus we receive that the hybrid methods have some unknown properties and therefore investigation of those methods are interesting.

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