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# Fuzzy Semantic Models of Fuzzy Concepts in Fuzzy Systems

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**Abstract—The fuzzy properties of language semantics are a central problem towards machine-enabled natural language processing in cognitive linguistics, fuzzy systems, and computational linguistics. A formal method for rigorously describing and manipulating fuzzy semantics is sought for bridging the gap between humans and cognitive fuzzy systems. The mathematical model of fuzzy concepts is rigorously described as a hyperstructure of fuzzy sets of attributes, objects, relations, and qualifications, which serves as the basic unit of fuzzy semantics for denoting languages entities in semantic analyses. The formal fuzzy concept is extended to complex structures where fuzzy modifiers and qualifiers are considered. An algebraic approach is developed to manipulate composite fuzzy semantic as a deductive process from a fuzzy concept to the determined semantics. The denotational mathematical structure of fuzzy semantic inference not only explains the fuzzy nature of human semantics and its comprehension, but also enables cognitive machines and fuzzy systems to mimic the human fuzzy inference mechanisms in cognitive linguistics, cognitive computing, and computational intelligence.**

**Keywords—Fuzzy systems, fuzzy semantics, fuzzy concept, cognitive linguistics, fuzzy inference, mathematical models of semantics, cognitive informatics, cognitive computing, soft computing, abstract intelligence, computational intelligence.**

## I. INTRODUCTION

FUZZY *semantics comprehension* and *fuzzy inference* are two of the central abilities of human brains that play a crucial role in thinking, perception, and problem solving [1, 2, 12, 13, 15, 17, 18, 23, 24, 25, 26, 28, 29, 31, 32, 35, 36]. *Semantics* in linguistics represents the meaning or the intension and extension of a language entity [3, 4, 7, 9, 10, 14]. *Formal semantics* [8, 9, 11, 19, 22, 27] focus on mathematical models for denoting meanings of symbols, concepts, functions, and behaviors, as well as their relations, which can be deduced onto a set of known concepts and behavioral processes in cognitive linguistics [5, 6, 28]. An *inference* is a cognitive process that deduces a proposition, particularly a causation, based on logical relations.

The taxonomy of semantics in natural languages can be classified into three categories [3, 4, 9, 14, 28, 32] known as those of entities (noun and noun phrases), behaviors (verbs and verb phrases), and modifiers (adjectives, adverbs, and related phrases). Semantics can also be classified into the categories of *to-be, to-have,* and *to-do* semantics [27]. A *to-be* semantics infers the meaning of an equivalent relation between an unknown and a known entity or concept. A *to-have* semantics denotes the meaning of a possessive structure or a composite entity. A *to-do* semantics embodies the process of a behavior or an action in a 5-dimensional behavioral space [22, 27].

The fuzzy nature of linguistic semantics as well as its cognition stems from inherent semantic ambiguity, context variability, and individual perceptions influenced by heterogeneous knowledge bases. Almost all problems in natural language processing and semantic analyses are constrained by these fundamental issues. Lotfi A. Zadeh extended methods for inferences to fuzzy sets and fuzzy logic [30, 33, 37], which provide a mathematical means for dealing with uncertainty and imprecision in reasoning, qualification, and quantification, particularly where vague linguistic variables are involved. Fuzzy inferences based on fuzzy sets are navel denotational mathematical means for rigorously dealing with degrees of matters, uncertainties, and vague semantics of linguistic entities, as well as for precisely reasoning the semantics of fuzzy causations. Typical fuzzy inference rules are those of fuzzy argument, implementation, deduction, induction, abduction, and analogy [22, 26, 32, 37].

This paper presents a theory of fuzzy concepts and fuzzy semantics for formal semantic manipulation in fuzzy systems and cognitive linguistics. The mathematical model of abstract fuzzy concepts is introduced in Section 2, which serves as the basic unit of fuzzy semantics in natural languages. A fuzzy concept is modeled as a fuzzy hyperstructure encompassing the fuzzy sets of attributes, objects, relations, and qualifications. Based on the mathematical model of fuzzy concepts, fuzzy semantic comprehension is reduced to a deduction process by algebraic operations on the fuzzy semantics. The mathematical model of fuzzy concept is extended to complex ones in Section 3 where fuzzy qualifiers are involved to modify fuzzy concepts. Algebraic operations on composite fuzzy semantics deduce the fuzzy semantics of a composite fuzzy concept to determined implications. The denotational mathematical structures of fuzzy semantic inference are elaborated by real-world examples in order to demonstrate their applications in cognitive linguistics, fuzzy

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systems, cognitive computing, and computational intelligence [6, 8, 12, 16, 20, 21].

## II. Fuzzy Semantics of Concepts in Fuzzy Inferences

The semantics of an entity in a natural language is used to be vaguely represented by a noun or noun phrase. In order to rigorously express the intension and extension of an entity expressed by a word, the noun entities can be formally described by an abstract concept in *concept algebra* [19] and *semantic algebra* [27]. An abstract concept is a cognitive unit to identify and model a concrete entity in the physical world or an abstract object in the perceived world, which can be formally described as follows.

**Definition 1.** Let $\mathcal{O}$ denote a finite nonempty set of *objects*, and $\mathcal{A}$ be a finite nonempty set of *attributes*. The *semantic discourse* $\mathfrak{U}_s$ is a triple, i.e.:

$$\begin{aligned}\mathcal{U}_s &= (\mathcal{O}, \mathcal{A}, \mathcal{R}) \\ &= \mathcal{R}: \mathcal{O} \to \mathcal{O} \mid \mathcal{O} \to \mathcal{A} \mid \mathcal{A} \to \mathcal{O} \mid \mathcal{A} \to \mathcal{A}\end{aligned} \quad (1)$$

where $\mathcal{R}$ is a set of relations between $\mathcal{O}$ and $\mathcal{A}$.

On the basis of the *semantic discourse*, a formal fuzzy concept can be defined as a certain composition of subsets of the three kinds of elements known as the objects, attributes, and relations.

**Definition 2.** A *fuzzy concept* $\widetilde{C}$ is a hyperstructure of language entities denoted by a 5-tuple encompassing the fuzzy sets of attributes $\widetilde{A}$, objects $\widetilde{O}$, internal relations $\widetilde{R^i}$, external relations $\widetilde{R^o}$, and qualifications $\widetilde{Q}$ i.e.:

$$\widetilde{C} \triangleq (\widetilde{A}, \widetilde{O}, \widetilde{R^i}, \widetilde{R^o}, \widetilde{Q}) \quad (2)$$

where

- $\widetilde{A}$ is a fuzzy set of *attributes* as the *intension* of the concept $\widetilde{C}$:

$$\widetilde{A} = \{(a_1, \mu(a_1)), (a_2, \mu(a_2)), \ldots, (a_n, \mu(a_n))\} \subseteq Þ\mathfrak{A} \quad (3)$$

where $Þ\mathfrak{A}$ denotes a power set of $\mathcal{A}$.

- $\widetilde{O}$ is a fuzzy set of *objects* as the *extension* of the concept $\widetilde{C}$:

$$\widetilde{O} = \{(o_1, \mu(o_1)), (o_2, \mu(o_2)), \ldots, (o_m, \mu(o_m))\} \subseteq Þ\mathfrak{O} \quad (4)$$

- $\widetilde{R^i}$ is a fuzzy set of *internal relations* between the fuzzy sets of objects $\widetilde{O}$ and attributes $\widetilde{A}$:

$$\begin{aligned}\widetilde{R^i} &= \widetilde{O} \times \widetilde{A} \subseteq Þ\mathfrak{R} \\ &= \mathop{R}_{j=1}^{|\widetilde{O}|} \mathop{R}_{i=1}^{|\widetilde{A}|} ((o_j, a_i), \mu(o_i) \bullet \mu(a_j))\end{aligned} \quad (5)$$

where the *big-R notation* [18, 25] expresses the Cartesian product of a series of repeated cross operations between $o_j$ and $a_i$, $1 \leq j \leq m$ and $1 \leq i \leq n$, which results in all the $(o_j, a_i)$ pairs.

- $\widetilde{R^o}$ is a fuzzy set of *external relations* between the fuzzy concept $\widetilde{C}$ and all potential ones $\widetilde{C'}$ in a knowledge base in $\mathfrak{U}_s$:

$$\begin{aligned}\widetilde{R^o} &= \widetilde{C} \times \widetilde{C'} \subseteq Þ\mathrm{R}, C' \neq C \wedge C' \sqsubseteq \mathfrak{U}_s \\ &= \mathop{R}_{k=1}^{|\mathcal{O}|} \{(\widetilde{C}, \widetilde{C'_k}), \mu(\widetilde{R}_k) = \sigma)\}\end{aligned} \quad (6)$$

where $\widetilde{C'}$ is a fuzzy set of external concepts in $\mathfrak{U}_s$, and the membership $\mu(\widetilde{R}_k)$ is determined by the *conceptual equivalency* σ between the sets of fuzzy attributes from each fuzzy concepts, i.e.:

$$\sigma = \frac{|\widetilde{A} \cap \widetilde{A'}|}{|\widetilde{A} \cup \widetilde{A'}|} \quad (7)$$

- $\widetilde{Q}$ is a fuzzy set of *qualifications* that modifies the concept $\widetilde{C}$ by weights in (0, 1] as a special part of the external relations $\widetilde{R^o}$:

$$\widetilde{Q} = \{(q_1, \omega(q_1)), (q_2, \omega(q_2)), \ldots, (q_p, \omega(q_p)))\} \subseteq Þ\mathfrak{R} \quad (8)$$

where $\widetilde{Q}$ is initially empty when the concept is independent. However, it obtains qualified properties and weights when the fuzzy concept is modified by an adjective or adjective phrase, or it is comparatively evaluated with other fuzzy concepts.

In the fuzzy concept model, Eqs. 5 and 6 denote general internal and external relations, respectively. A concrete fuzzy relation in a specific fuzzy concept will be an instantiation of the general relations tailored by a given characteristic matrix on the Cartesian products.

As described in Definition 2, the important properties of a formal fuzzy concept are the fuzzy set of essential attributes as its *intension*; the fuzzy set of instantiated objects as its *extension*; and the adaptive capability to autonomously interrelate the concept to other concepts in an existing knowledge base in $\mathfrak{U}_s$.

**Example 1.** A fuzzy concept 'pen', $\widetilde{C}(pen)$, can be formally described according to Definition 2 as follows:

$$\begin{aligned}\widetilde{C}(pen) &\triangleq \widetilde{C}(\widetilde{A}, \widetilde{O}, \widetilde{R^i}, \widetilde{R^o}, \widetilde{Q}) \\ &= \widetilde{pen}((\widetilde{A}, \widetilde{\mu_C(A)}), (\widetilde{O}, \widetilde{\mu_C(O)}), \widetilde{R^{i'}}, \widetilde{R^{o'}}, \widetilde{Q}) \\ &= \begin{cases} \widetilde{A} = \{(a_1, \mu(a_1)), (a_2, \mu(a_2)), (a_3, \mu(a_3)), (a_4, \mu(a_4))\} \\ \quad = \{(writing\_tool,\ 1.0),\ (ink,\ 0.9),\ (nib,\ 0.9), \\ \qquad (ink\_container,\ 0.8)\} \\ \widetilde{O} = \{(o_1, \mu(o_1)), (o_2, \mu(o_2)), (o_3, \mu(o_3)), (o_4, \mu(o_4))\} \\ \quad = \{(ballpo\mathrm{int},\ 1.0),\ (fountain,\ 1.0), \\ \qquad (pencil,\ 0.9),\ (brush,\ 0.7)\} \\ \widetilde{R^i} = \widetilde{O} \times \widetilde{A} \\ \widetilde{R^o} = \widetilde{C} \times \widetilde{C'} \\ \widetilde{Q} = \varnothing \end{cases}\end{aligned}$$

**Example 2.** A fuzzy concept 'man', $\tilde{C}(man)$, can be formally described based on Definition 4 as follows:

$$\begin{aligned}\tilde{C}(man) &\triangleq \tilde{C}(\tilde{A},\tilde{O},\widetilde{R^i},\widetilde{R^o},\tilde{Q}) \\ &= \widetilde{man}((\tilde{A},\widetilde{\mu_C(A)}),(\tilde{O},\widetilde{\mu_C(O)}),\widetilde{R^{i'}},\widetilde{R^{o'}},\tilde{Q}) \\ &= \begin{cases}\tilde{A} = \{(human\_being,\ 1.0),\ (male,\ 1.0),\ (adult,\ 0.9)\} \\ \tilde{O} = \{(American,\ 1.0),\ (Australia,\ 1.0), \\ \quad (business\_man,\ 1.0),\ ...\} \\ \widetilde{R^{i'}} = \tilde{O}\times\tilde{A} \\ \widetilde{R^{o'}} = \tilde{C}\times\widetilde{C'} \\ \tilde{Q} = \varnothing\end{cases}\end{aligned}$$

Applying the fuzzy concept model as a basic unit of semantic knowledge in $\mathfrak{U}_s$, the fuzzy semantics in natural languages can be expressed as a mapping from a fuzzy language entity to a determined fuzzy concept where its sets of fuzzy attributes, objects, relations, and qualifications are specified.

**Definition 3.** The *fuzzy semantics of an entity e*, $\tilde{\Theta}(e)$, is an equivalent fuzzy concept $\tilde{C}_e$ in $\mathfrak{U}_s$, i.e.:

$$\begin{aligned}\tilde{\Theta}(e) &\triangleq \tilde{\Theta}(e = \widetilde{C_e}) \\ &= \widetilde{C}_e(\widetilde{A_e},\widetilde{O_e},\widetilde{R_e^i},\widetilde{R_e^o},\widetilde{Q_e})\end{aligned} \tag{9}$$

where $\tilde{C}_e$ is denoted according to the generic model of fuzzy concepts as given in Definition 2.

**Example 3.** The fuzzy semantics of a language entity 'pen', denoted by $\Theta(\tilde{C}(pen))$, can be formally derived according to Definition 3 and Example 1 as follows:

$$\begin{aligned}\widetilde{\Theta_e}(pen) &\triangleq \tilde{\Theta}(e = \tilde{C}(pen)) \\ &= \widetilde{pen}((\tilde{A},\widetilde{\mu_{\tilde{C}}(A)}),(\tilde{O},\widetilde{\mu_{\tilde{C}}(O)}),\widetilde{R^i},\widetilde{R^o},\tilde{Q}) \\ &= \begin{cases}\tilde{A} = \{(writing\_tool,\ 1.0),\ (ink,\ 0.9),\ (nib,\ 0.9), \\ \quad (ink\_container,\ 0.8)\} \\ \tilde{O} = \{(ballpo\mathrm{int},\ 1.0),\ (fountain,\ 1.0), \\ \quad (pencil,\ 0.9),\ (brush,\ 0.7)\} \\ \widetilde{R^i} = \tilde{O}\times\tilde{A} \\ \widetilde{R^o} = \tilde{C}\times\widetilde{C'} \\ \tilde{Q} = \varnothing\end{cases}\end{aligned}$$

**Example 4.** Similarly, the fuzzy semantics of a language entity 'man', denoted by $\Theta(\tilde{C}(man))$, can be formally derived based on Definition 3 and Example 2 as follows:

$$\begin{aligned}\widetilde{\Theta_e}(man) &\triangleq \tilde{\Theta}(e = \tilde{C}(man)) \\ &= \widetilde{man}((\tilde{A},\mu_{\tilde{C}}(A)),(\tilde{O},\mu_{\tilde{C}}(O)),\widetilde{R^{i'}},\widetilde{R^{o'}},\tilde{Q}) \\ &= \begin{cases}\tilde{A} = \{(human\_being,\ 1.0),\ (male,\ 1.0),\ (adult,\ 0.9)\} \\ \tilde{O} = \{(American,\ 1.0),\ (Australia,\ 1.0), \\ \quad (business\_man,\ 1.0),\ ...\} \\ \widetilde{R^{i'}} = \tilde{O}\times\tilde{A} \\ \widetilde{R^{o'}} = \tilde{C}\times\widetilde{C'} \\ \tilde{Q} = \varnothing\end{cases}\end{aligned}$$

Therefore, on the basis of Definition 3, fuzzy semantic analyses and comprehension in natural languages can be formally described as a deductive process from a fuzzy entity to a determined fuzzy concept.

**Corollary 1.** The *rule of semantic deduction* states that the semantics of a given fuzzy entity is comprehended in semantic analysis, if and only if its fuzzy semantics can be reduced onto a known fuzzy concept with determined membership and weight values.

## III. FUZZY SEMANTICS OF MODIFIERS ON CONCEPTS IN FUZZY INFERENCES

The semantics of fuzzy concepts is usually modified by an adjective or an adjective phrase in language expressions in order to fine tune its qualification such as degree, scope, quality, constraint, purpose, and etc. Therefore, the fuzzy semantics of fuzzy concepts as developed in Section 2 can be extended to deal with composite semantics of noun phrases modified by determiners and degree words [19, 27, 28, 31, 32, 34].

The modifier in cognitive linguistics is words or phrases that elaborate, limit, and qualify a noun or noun phrase in the categories of determiners, qualifiers, degrees, and negations [6, 28]. A fuzzy modifier can be represented as a fuzzy set with certain weights of memberships [31, 32, 34]. For instance, Zadeh considered the fuzzy effects of some special adverbs on adjectives such as 'very, very', 'very little', 'positive', and 'negative' in 1975, which were modeled as nonlinear exponential weights on the target adjectives [32]. However, the general semantics relations between a fuzzy linguistic entity (noun) and its fuzzy modifier (adverb-adjective phrase) are yet to be studied.

**Definition 4.** A *fuzzy modifier* $\tilde{\tau}$ is a special fuzzy set that represents an adjective or adjective phrase in natural languages where its memberships are replaced by intentional weights of the modifier, $\omega_{\tilde{\tau}}(\tau_k)$, $1 \le k \le z$, and *z* is a constant, i.e.:

$$\begin{aligned}\tilde{\tau} &= \{\mathop{R}_{k=1}^{z}(\tau_k, \omega_{\tilde{\tau}}(\tau_k))\},\ \omega_{\tilde{\tau}}(\tau_k) \in (0,1] \\ &= \{(\tau_1,\omega(\tau_1)),(\tau_2,\omega(\tau_2)),...,(\tau_z,\omega(\tau_z)\}\end{aligned} \tag{10}$$

where the weights of $\tilde{\tau}$ is normalized in the domain $(0,1]$.

**Example 5.** A fuzzy modifier 'good' on the *quality* of a fuzzy entity can be formally described as a fuzzy set $\tilde{\tau}(good)$ according to Definition 4 as follows:

$$\begin{aligned}\tilde{\tau}(good) &= \{\mathop{R}_{k=1}^{4}(\tau_k(quality),\ \mu_{\tilde{\tau}}(\tau_k))\} \\ &= \{(neutral,\ 0.1),\ (ok,\ 0.6), \\ &\quad (excellent,\ 0.8),\ (perfect,\ 1.0)\}\end{aligned} \tag{11}$$

**Example 6.** A fuzzy modifier 'old' on the fuzzy entity *ages* can be formally described as a fuzzy set $\tilde{\tau}(old)$ as follows:

$$\begin{aligned}\tilde{\tau}(old) &= \{\mathop{R}_{k=1}^{6}(\tau_k(age),\ \omega_{\tilde{\tau}}(\tau_k))\} \\ &= \{([\text{1-20}],\ 0),\ ([\text{21-30}],\ 0.1),\ ([\text{31-50}],\ 0.4), \\ &\quad ([\text{51-65}],\ 0.7),\ ([\text{66-80}],\ 0.9),\ ([>80],\ 1.0)\}\end{aligned} \tag{12}$$

**Definition 5.** A *fuzzy qualifier* $\tilde{\delta}$ is a *special* fuzzy set of degree adverbs or adverb phrases to modify $\tilde{\tau}$ in natural languages where their memberships are replaced by *intentional weights* of degree and extends, $\omega_{\tilde{\delta}_l}(\delta_l),\ 1 \le l \le q,\ \text{and } q \text{ is a constant}$, i.e.:

$$\begin{aligned}\tilde{\delta} &= \{\mathop{R}_{l=1}^{p}(\delta_l, \omega_{\tilde{\delta}}(\delta_l))\},\ \omega_{\tilde{\delta}}(\delta_l) \in \pm(0,3] \\ &= \{(\delta_1,\omega(\tau_1)),(\delta_2,\omega(\delta_2)),...,(\delta_p,\omega(\delta_p)\}\end{aligned} \tag{13}$$

where the weights of $\tilde{\delta}$ is constrained in the domain $\pm[1,3]$ corresponding to the *neutral* (1), *comparative* (2), and *superlative* (3) degrees of adverbs and adjectives in natural languages.

**Example 7.** A typical fuzzy set of qualifiers, $\tilde{\delta}$, can be described according to Definition 5 as follows:

$$\begin{aligned}\tilde{\delta} = \{&(definitely\_not,\ \text{-}3.0),\ (imperfectly,\ \text{-}1.5), \\ &(neutral\_negative,\ \text{-}1.0),\ (somewhat,\ 0.5), \\ &(fairly,\ 1.2),\ (quite,\ 1.5),\ (excellently,\ 2.0), \\ &(extremely,\ 3.0)\},\ \omega_{\tilde{\delta}}(\delta) \in \pm(0,3]\end{aligned} \tag{14}$$

**Definition 6.** A *composite fuzzy modifier* $\widetilde{\delta\tau}$ is a product of a fuzzy qualifier $\tilde{\delta}$ and a fuzzy modifier $\tilde{\tau}$. The value of the composite modifiers is determined by the product of their weights, i.e.:

$$\begin{aligned}\Theta(\tilde{\delta} \bullet \tilde{\tau}) &\triangleq \Theta(\widetilde{\delta(y) \bullet \tau(x)}) \\ &= \omega_{\tilde{\delta}}(y) \bullet \omega_{\tilde{\tau}}(x),\ 0 < \omega_{\tilde{\tau}}(x) \le 1, \\ &\quad \text{-}3 \le \omega_{\tilde{\delta}}(y) \le 3,\ \omega_{\tilde{\delta}}(y) \ne 0\end{aligned} \tag{15}$$

where the *combined domain* of composite modifiers is $\pm(0,3]$ in order to be consistent to the modifiers in real-world languages.

In case a weight of the fuzzy qualifiers is less than zero, the composite modifier represents a negative intention. For instance, $\tilde{\delta} \bullet \tilde{\tau} = \tilde{\delta}(neutral\_negative) \bullet \tilde{\tau}(good)$ implies a weight of qualification in the semantics as $\omega(\tilde{\delta}(neutral\_negative)) \bullet \omega(\tilde{\tau}(good)) = -1 \bullet 0.6 = -0.6.$

On the basis of the formal semantics of fuzzy modifiers $\tilde{\tau}$, qualifiers $\tilde{\delta}$, and composite modifiers $\widetilde{\tau'} = \widetilde{\delta\tau}$, the composite fuzzy semantics of language entities modified by $\widetilde{\delta\tau}$ can be quantitatively expressed.

**Definition 7.** The *composite fuzzy semantic* of a fuzzy concept $\widetilde{C}$ qualified by a *fuzzy* modifier $\tilde{\tau}$, qualifier $\tilde{\delta}$, and/or a composite fuzzy modifier $\widetilde{\tau'} = \widetilde{\delta\tau}$, denoted by $\tilde{\Theta}(\widetilde{C'}) = \tilde{\Theta}(\widetilde{\tau'} \bullet \widetilde{C})$, is a complex semantics of the fuzzy concept $\widetilde{C}$ qualified by a certain weight of the composite modifier, i.e.:

$$\begin{aligned}\tilde{\Theta}(\widetilde{\tau'} \bullet \widetilde{C}) &\triangleq \tilde{\Theta}(\widetilde{C} \mid \widetilde{Q} = \widetilde{\delta\tau}) \\ &= \widetilde{C}(\tilde{A},\tilde{O},\widetilde{R^i},\widetilde{R^o},\tilde{Q} \mid \tilde{Q} = \widetilde{\delta\tau}) \\ &= \widetilde{C}((\tilde{A},\widetilde{\mu_{\tilde{A}}(A)}),(\tilde{O},\widetilde{\mu_{\tilde{O}}(O)}),\widetilde{R^i},\widetilde{R^o},(\tilde{Q} = \widetilde{\delta\tau})) \\ &= \widetilde{C}(\tilde{A},\tilde{O},\widetilde{R^i},\widetilde{R^o},\tilde{Q})\end{aligned} \tag{16}$$

where the fuzzy set of composite modifiers imposes a specific set of weights of intentional qualifications $\widetilde{Q}$ in the modified semantics of the target fuzzy concept, and $\tilde{\delta} = 1$ if it is absent.

**Example 8.** Given a fuzzy concept $\widetilde{C}(pen)$ as obtained in Example 1, the composite semantics $\tilde{\Theta}(\tilde{\tau} \bullet \widetilde{C}) = \tilde{\Theta}(excellent\_pen)$ qualified by the fuzzy modifier $\tilde{\tau}(good)$ can be determined according to Definition 7, i.e.:

$$\begin{aligned}&\tilde{\Theta}(\widetilde{\delta\tau} \bullet \widetilde{C}) = \tilde{\Theta}(\tilde{\tau}(good) \bullet \widetilde{pen}) \\ &= \widetilde{excellent\_pen}((\tilde{A},\widetilde{\mu_{\tilde{A}}(A)}),(\tilde{O},\widetilde{\mu_{\tilde{O}}(O)}),\widetilde{R^i},\widetilde{R^o}, \\ &\qquad (\tilde{Q} = \tau_0 \mid \tau_0 = \tau(excellent) = 0.8)) \\ &= \begin{cases}\tilde{A} = \{(writing,\ 1.0),\ (ink,\ 0.9),\ (nib,\ 0.9), \\ \quad (ink\_container,\ 0.8)\} \\ \tilde{O} = \{(ballpoint,\ 1.0),\ (fountain,\ 1.0),\ (pencil,\ 0.9), \\ \quad (brush,\ 0.7)\} \\ \widetilde{R^i} = \tilde{O} \times \tilde{A} \\ \widetilde{R^o} = \widetilde{C} \times \widetilde{C'} \\ \tilde{Q}(excellent) = 0.8\end{cases}\end{aligned}$$

**Example 9.** The fuzzy semantics of $\widetilde{C}(excellent\_pen)$ obtained in Example 8 may be further modified by a qualifier $\tilde{\delta}(extremely)$ that results in $\widetilde{C}(extremely\_excellent\_pen)$ as follows:

$$\begin{aligned}
&\widetilde{\Theta}(\widetilde{\delta\tau} \bullet \widetilde{C}) = \widetilde{\Theta}(\tilde{\delta}(extremely) \bullet \tilde{\tau}(good) \bullet \widetilde{pen}) \\
&= \widetilde{extremely_excellent_pen}((\widetilde{A}, \widetilde{\mu_{\tilde{A}}(A)}), (\widetilde{O}, \widetilde{\mu_{\tilde{O}}(O)}), \widetilde{R^i}, \widetilde{R^o}, \\
&\quad (\widetilde{Q} = \tau'_0 \mid \tau'_0 = \tilde{\delta}(extremely) \bullet \tau(excellent) = 3.0 \bullet 0.8)) \\
&= \begin{cases}
\widetilde{A} = \{(writing_tool,\ 1.0),\ (ink,\ 0.9),\ (nib,\ 0.9), \\
\qquad (ink_container,\ 0.8)\} \\
\widetilde{O} = \{(ballpoint,\ 1.0),\ (fountain,\ 1.0),\ (pencil,\ 0.9), \\
\qquad (brush,\ 0.7)\} \\
\widetilde{R^i} = \widetilde{O} \times \widetilde{A} \\
\widetilde{R^o} = \widetilde{C} \times \widetilde{C}' \\
\widetilde{Q}(extremely_excellent) = 2.4
\end{cases}
\end{aligned}$$

**Example 10.** Given a fuzzy concept $\widetilde{C}(man)$ as described in Example 2, the composite semantics $\widetilde{\Theta}(\tilde{\tau} \bullet \widetilde{C}) = \widetilde{\Theta}(old\_man)$ qualified by the fuzzy modifier $\tilde{\tau}$ (*old*) can be determined according to Definition 7 as follows:

$$\begin{aligned}
&\widetilde{\Theta}(\tilde{\tau} \bullet \widetilde{C}) = \widetilde{\Theta}(\tilde{\tau}(old) \bullet \widetilde{man}),\ \tau_0 = \tau(60) = 0.7 \\
&\quad = \widetilde{old_man}((\widetilde{A}, \widetilde{\mu_{\tilde{A}}(A)}), (\widetilde{O}, \widetilde{\mu_{\tilde{O}}(O)}), \widetilde{R^i}, \widetilde{R^o}, (\widetilde{Q} = \tau_0)) \\
&\quad = \begin{cases}
\widetilde{A} = \{(human_being,\ 1.0),\ (male,\ 1.0), \\
\qquad (adult,\ 0.9)\} \\
\widetilde{O} = \{(American,\ 1.0),\ (Australia,\ 1.0), \\
\qquad (business_man,\ 1.9),\ ...\} \\
\widetilde{R^i} = \widetilde{O} \times \widetilde{A} \\
\widetilde{R^o} = \widetilde{old_man} \times \widetilde{C}' \\
\widetilde{Q}(old) = 0.7
\end{cases}
\end{aligned}$$

Example 10 indicates that, against the fuzzy qualifier $\tilde{\tau}(old)$, a man in the age of 60 is 0.7 (*quite likely*) as an old man. Similarly, other instantiations modified by the qualifiers may denote that a man in the age of 25 is 0.1 (*unlikely*) as an old man; and a man in the age of 85 is 1.0 (*definitely*) as an old man.

**Example 11.** The fuzzy semantics of $\widetilde{C}(old\_man)$ obtained in Example 10 may be further modified by a qualifier $\tilde{\delta}(quite)$ that results in $\widetilde{C}(a\_quite\_old\_man)$ as follows:

$$\begin{aligned}
&\widetilde{\Theta}(\widetilde{\delta\tau} \bullet \widetilde{C}) = \widetilde{\Theta}(\tilde{\delta}(quite) \bullet \tilde{\tau}(old) \bullet \widetilde{man}) \\
&\quad = \widetilde{a_quite_old_man}((\widetilde{A}, \widetilde{\mu_{\tilde{A}}(A)}), (\widetilde{O}, \widetilde{\mu_{\tilde{O}}(O)}), \widetilde{R^i}, \widetilde{R^o}, \\
&\qquad (\widetilde{Q} = \tau'_0 \mid \tau'_0 = \tilde{\delta}(quite) \bullet \tau(60) = 1.5 \bullet 0.7)) \\
&= \begin{cases}
\widetilde{A} = \{(human_being,\ 1.0),\ (male,\ 1.0),\ (adult,\ 0.9)\} \\
\widetilde{O} = \{(American,\ 1.0),\ (Australia,\ 1.0), \\
\qquad (business_man,\ 1.0),\ ...\} \\
\widetilde{R^i} = \widetilde{O} \times \widetilde{A} \\
\widetilde{R^o} = \widetilde{old_man} \times \widetilde{C}' \\
\widetilde{Q}(quite_old) = 1.05
\end{cases}
\end{aligned}$$

The fuzzy nature of language semantics and their comprehension is formally explained by the mathematical models of fuzzy concepts and fuzzy semantics qualified by fuzzy modifiers. Based on the formal theory, fuzzy semantic inferences can be rigorously manipulated to deal with fuzzy degrees of matters, uncertainties, vague semantics, and fuzzy causality in cognitive linguistics and fuzzy systems. This work enables cognitive machines, cognitive robots, and fuzzy system to mimic the human intelligent ability and the cognitive processes in cognitive linguistics, fuzzy inferences, cognitive computing, and computational intelligence.

## IV. Conclusion

The mathematical models of fuzzy concepts and fuzzy semantics have provided a formal explanation for the fuzzy nature of human language processing and real-time semantics interpretation. It has been identified that the basic unit of linguistic entities that carries unique and unambiguous semantics is a fuzzy concept, which can be modeled as a fuzzy hyperstructure encompassing fuzzy sets of attributes, objects, relations, and qualifications. Complex fuzzy concepts in natural languages have been modeled as a composite fuzzy concept where fuzzy qualifiers are involved to modify the fuzzy concept. As a result, the fuzzy semantics of composite fuzzy concepts has been denoted as algebraic operations on the fuzzy qualification of the target fuzzy concept. This work has demonstrated that fuzzy semantic comprehension is a deductive process, where complex fuzzy semantics can be formally expressed by algebraic operations on elementary ones with fuzzy modifiers. The denotational mathematical structure of fuzzy concepts and fuzzy semantics not only reveals the fuzzy properties of human semantic comprehension, but also enables cognitive machines and fuzzy systems to mimic the human fuzzy inference mechanisms in cognitive linguistics, fuzzy systems, cognitive computing, and computational intelligence.

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